

WEIGHTED PROJECTIVE LINES AND RATIONAL SURFACE SINGULARITIES

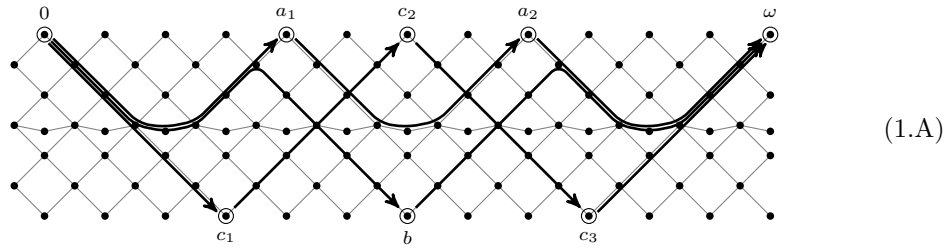
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ABSTRACT. In this paper we study rational surface singularities R with star shaped dual graphs, and under very mild assumptions on the self-intersection numbers we give an explicit description of all their special Cohen–Macaulay modules. We do this by realising R as a certain \mathbb{Z} -graded Veronese subring $S^{\vec{x}}$ of the homogeneous coordinate ring S of the Geigle–Lenzing weighted projective line \mathbb{X} , and we realise the special CM modules as explicitly described summands of the canonical tilting bundle on \mathbb{X} . We then give a second proof that these are special CM modules by comparing $\mathbf{qgr} S^{\vec{x}}$ and $\mathbf{coh} \mathbb{X}$, and we also give a necessary and sufficient combinatorial criterion for these to be equivalent categories. In turn, we show that $\mathbf{qgr} S^{\vec{x}}$ is equivalent to $\mathbf{qgr} \Gamma$ where Γ is the corresponding reconstruction algebra, and that the degree zero piece of Γ coincides with Ringel’s canonical algebra. This implies that Γ contains the canonical algebra and furthermore $\mathbf{qgr} \Gamma$ is derived equivalent to the canonical algebra, thus linking the reconstruction algebra of rational surface singularities to the canonical algebra of representation theory.

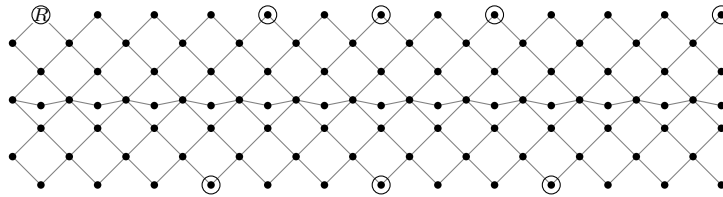
1. INTRODUCTION

1.1. Motivation and Overview. It is well known that any rational surface singularity has only finitely many indecomposable special CM modules, but it is in general a difficult task to classify and describe them explicitly. In this paper we use the combinatorial structure encoded in the homogeneous coordinate ring S of the Geigle–Lenzing weighted projective line \mathbb{X} to solve this problem for a large class of examples arising from star shaped dual graphs, extending our previous work [IW] to cover a much larger class of varieties. In the process, we link S , its Veronese subrings, the reconstruction algebra and the canonical algebra, through a range of categorical equivalences.

A hint of a connection between rational surface singularities and the canonical algebra can be found in the lecture notes [R2]. In his study of the representation theory of the canonical algebra $\Lambda_{\mathbf{p}, \lambda}$, Ringel drew pictures [R2, p196] including the following one for type \tilde{E}_7 :



What is remarkable is that this picture is identical to one the authors drew in [IW, 8.2] when classifying special CM modules for a certain family of quotient singularities $\mathbb{C}[[x, y]]^G$ with $G \leq \mathrm{GL}(2, k)$, namely



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Further, although they were not drawn in [IW], the arrows in (1.A) are implicit in the calculation of the quiver of the corresponding reconstruction algebra [W2, §4]. This paper grew out of trying to give a conceptual explanation for this coincidence, since a connection between the mathematics underpinning the two pictures did not seem to be known.

In fact, the connection turns out to be explained by a very general phenomenon. Recall first that one of the basic properties of the canonical algebra $\Lambda_{\mathbf{p}, \lambda}$ is that there is always a derived equivalence [GL1]

$$D^b(\text{coh } \mathbb{X}_{\mathbf{p}, \lambda}) \simeq D^b(\text{mod } \Lambda_{\mathbf{p}, \lambda})$$

where $\text{coh } \mathbb{X}_{\mathbf{p}, \lambda}$ is the weighted projective line of Geigle–Lenzing (for details, see §1.2). Thus, to explain the above coincidence, we are led to consider the possibility of linking the weighted projective line, viewed as a Deligne–Mumford stack, to the study of rational surface singularities. However, the weighted projective line $\mathbb{X}_{\mathbf{p}, \lambda}$ cannot itself be the stack that we are after, since it only has dimension one, and rational surface singularities have, by definition, dimension two.

We need to increase the dimension, and the most naive way to do this is to consider the total space of a line bundle over $\mathbb{X}_{\mathbf{p}, \lambda}$. We thus choose any member of the grading group $\vec{x} \in \mathbb{L}$ and consider the total space stack $\text{Tot}(\mathcal{O}_{\mathbb{X}}(-\vec{x}))$ (for definition, see §1.2). From tilting on this and its coarse moduli, under mild assumptions we prove that the Veronese subring $S^{\vec{x}} := \bigoplus_{i \in \mathbb{Z}} S_{i\vec{x}}$ is a weighted homogeneous rational surface singularity, giving the first concrete connection between the above two settings. Furthermore, from the stack $\text{Tot}(\mathcal{O}_{\mathbb{X}}(-\vec{x}))$ we then describe the special CM $S^{\vec{x}}$ -modules, and give precise information regarding the minimal resolution of $\text{Spec } S^{\vec{x}}$ and its derived category. Using this, through a range of categorical equivalences we are then able to relate $\text{CM}^{\mathbb{Z}} S^{\vec{x}}$ and $\text{vect } \mathbb{X}$, which finally allows us to explain categorically why the above two pictures must be the same.

We now describe our results in detail.

1.2. Veronese Subrings and Special CM modules. Throughout, let k denote an algebraically closed field of characteristic zero. For any $n \geq 0$, choose positive integers p_1, \dots, p_n with all $p_i \geq 2$ and set $\mathbf{p} := (p_1, \dots, p_n)$. Furthermore, choose pairwise distinct points $\lambda_1, \dots, \lambda_n$ in \mathbb{P}^1 , and denote $\lambda := (\lambda_1, \dots, \lambda_n)$. Let $\ell_i(t_0, t_1) \in k[t_0, t_1]$ be the linear form defining λ_i , and write

$$S_{\mathbf{p}, \lambda} = S := \frac{k[t_0, t_1, x_1, \dots, x_n]}{(x_i^{p_i} - \ell_i(t_0, t_1) \mid 1 \leq i \leq n)}.$$

Moreover, let $\mathbb{L} = \mathbb{L}(p_1, \dots, p_n)$ denote the abelian group generated by the elements $\vec{x}_1, \dots, \vec{x}_n$ subject to the relations $p_1 \vec{x}_1 = p_2 \vec{x}_2 = \dots = p_n \vec{x}_n =: \vec{c}$. With this input S is an \mathbb{L} -graded algebra with $\deg x_i := \vec{x}_i$ and $\deg t_j := \vec{c}$, and \mathbb{L} is a rank one abelian group, possibly containing torsion. Often we normalize λ so that $\lambda_1 = 1$, $\lambda_2 = \infty$ and $\lambda_3 = 1$, however it is important for changing parameters later that we allow ourselves flexibility.

From this, consider the stack

$$\mathbb{X}_{\mathbf{p}, \lambda} = \mathbb{X} := [(\text{Spec } S_{\mathbf{p}, \lambda} \setminus 0) / \text{Spec } k\mathbb{L}],$$

with coarse moduli space denoted $X_{\mathbf{p}, \lambda} = X$. It is well known that $X \cong \mathbb{P}^1$, regardless of \mathbf{p} and λ (see 2.1(2)).

To increase dimension we then choose $0 \neq \vec{x} \in \mathbb{L}_+$ (for definition see 2.1(1)) and consider both the Veronese subring $S^{\vec{x}} := \bigoplus_{i \in \mathbb{Z}} S_{i\vec{x}}$ and the total space stack

$$\text{Tot}(\mathcal{O}_{\mathbb{X}_{\mathbf{p}, \lambda}}(-\vec{x})) := [(\text{Spec } S_{\mathbf{p}, \lambda} \setminus 0 \times \text{Spec } k[t]) / \text{Spec } k\mathbb{L}],$$

where \mathbb{L} acts on t with weight $-\vec{x}$. Writing $\vec{x} = \sum_{i=1}^n a_i \vec{x}_i + a \vec{c}$ in normal form (see 2.1(1)), we show in 3.5 that the coarse moduli space $T^{\vec{x}}$ is a surface containing a \mathbb{P}^1 , and on that \mathbb{P}^1 complete locally the singularities of $T^{\vec{x}}$ are of the form

$$\begin{array}{ccccccc} \lambda_1 & & \lambda_2 & & \lambda_3 & & \mathbb{P}^1 & & \lambda_n \\ \bullet & & \bullet & & \bullet & & & & \bullet \\ \frac{1}{p_1}(1, -a_1) & & \frac{1}{p_2}(1, -a_2) & & \frac{1}{p_3}(1, -a_3) & & \dots & & \frac{1}{p_n}(1, -a_n) \end{array} \quad (1.B)$$

where for notation see 2.10. There is a natural map $\psi: T^{\vec{x}} \rightarrow \text{Spec } S^{\vec{x}}$ (§3), and we let $\varphi: Y^{\vec{x}} \rightarrow T^{\vec{x}}$ denote the minimal resolution of $T^{\vec{x}}$. This datum can be summarized by the following commutative diagram

$$\begin{array}{ccccc}
 \text{Tot}(\mathcal{O}_{\mathbb{X}}(-\vec{x})) & \xrightarrow{q} & \mathbb{X} & & \\
 \downarrow g & & \downarrow f & & \\
 Y^{\vec{x}} & \xrightarrow{\varphi} & T^{\vec{x}} & \xrightarrow{p} & X \cong \mathbb{P}^1 \\
 \searrow \pi & & \downarrow \psi & & \\
 & & \text{Spec } S^{\vec{x}} & &
 \end{array} \tag{1.C}$$

We remark that the coarse moduli space $T^{\vec{x}}$ is a singular line bundle in the sense of Dolgachev [D, §4] and Pinkham [P, §3], which also appears in the work of Orlik–Wagreich [OW] and many others. However, the key difference in our approach is that the grading group giving the quotient is \mathbb{L} not \mathbb{Z} , and indeed it is the extra combinatorial structure of \mathbb{L} that allows us to extract the geometry much more easily.

Write $\mathcal{E} := \bigoplus_{i \in [0, \vec{c}]} \mathcal{O}_{\mathbb{X}}(i)$ for the Geigle–Lenzing tilting bundle on \mathbb{X} [GL1]. Our first main theorem is the following:

Theorem 1.1. *If $0 \neq \vec{x} = \sum_{i=1}^n a_i \vec{x}_i + a\vec{c} \in \mathbb{L}_+$, then with notation as in (1.C),*

- (1) (=3.3) *$p^*(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1))$ is a tilting bundle on $T^{\vec{x}}$. In particular $H^i(\mathcal{O}_{T^{\vec{x}}}) = 0$ for all $i \geq 1$, and $S^{\vec{x}}$ is a rational surface singularity.*
- (2) (=3.2) *$q^*\mathcal{E}$ is a tilting bundle on $T^{\vec{x}} := \text{Tot}(\mathcal{O}_{\mathbb{X}}(-\vec{x}))$ such that*

$$\begin{array}{ccc}
 \text{D}^b(\text{Qcoh } T^{\vec{x}}) & \xrightarrow[\sim]{\text{RHom}_{T^{\vec{x}}}(q^*\mathcal{E}, -)} & \text{D}^b(\text{Mod End}_{T^{\vec{x}}}(q^*\mathcal{E})) \\
 \text{R}q_* \downarrow & & \downarrow \text{res} \\
 \text{D}^b(\text{Qcoh } \mathbb{X}) & \xrightarrow[\sim]{\text{RHom}_{\mathbb{X}}(\mathcal{E}, -)} & \text{D}^b(\text{Mod } \Lambda_{\mathbf{p}, \lambda})
 \end{array}$$

commutes, where $\Lambda_{\mathbf{p}, \lambda}$ is the canonical algebra of Ringel.

- (3) (=3.13) *There is a fully faithful embedding $\text{D}^b(\text{coh } Y^{\vec{x}}) \hookrightarrow \text{D}^b(\text{coh } T^{\vec{x}})$.*
- (4) (=4.9) *If further $(p_i, a_i) = 1$ for all $1 \leq i \leq n$, then the embedding in (3) is an equivalence if and only if every a_i is 1, that is $\vec{x} = \sum_{i=1}^n \vec{x}_i + a\vec{c}$.*

The coprime assumption in 1.1(4) is not restrictive, since we show in §4.3 that we can always replace $\mathbb{X}_{\mathbf{p}, \lambda}$ by some equivalent $\mathbb{X}_{\mathbf{p}', \lambda'}$ for which the coprime assumption holds. See 4.12 for full details.

Combining the above tilting result with (1.B) and a combinatorial argument, we are in fact able to determine the precise dual graph (for definition see 2.3) of the morphism π in (1.C). Recall that for each $\frac{1}{p_i}(1, -a_i)$ in (1.B) with $a_i \neq 0$, we can consider the Hirzebruch–Jung continued fraction expansion

$$\frac{p_i}{p_i - a_i} = \alpha_{i1} - \frac{1}{\alpha_{i2} - \frac{1}{\alpha_{i3} - \frac{1}{\ddots}}}} := [\alpha_{i1}, \dots, \alpha_{im_i}], \tag{1.D}$$

with each $\alpha_{ij} \geq 2$; see §2.4 for full details.

Theorem 1.2 (=3.8, 4.21). *Let $0 \neq \vec{x} \in \mathbb{L}_+$ and as above write $\vec{x} = \sum_{i=1}^n a_i \vec{x}_i + a\vec{c}$ in normal form. Then the dual graph of the morphism $\pi: Y^{\vec{x}} \rightarrow \text{Spec } S^{\vec{x}}$ is*

$$\begin{array}{cccc}
 \bullet -\alpha_{1m_1} & \bullet -\alpha_{2m_2} & \bullet -\alpha_{3m_3} & \bullet -\alpha_{nm_n} \\
 \vdots & \vdots & \vdots & \vdots \\
 \bullet -\alpha_{1m_1-1} & \bullet -\alpha_{2m_2-1} & \bullet -\alpha_{3m_3-1} & \bullet -\alpha_{nm_n-1} \\
 \vdots & \vdots & \vdots & \vdots \\
 \bullet -\alpha_{12} & \bullet -\alpha_{22} & \bullet -\alpha_{32} & \bullet -\alpha_{n2} \\
 \vdots & \vdots & \vdots & \vdots \\
 \bullet -\alpha_{11} & \bullet -\alpha_{21} & \bullet -\alpha_{31} & \bullet -\alpha_{n1} \\
 & & \downarrow & \\
 & & \bullet -\beta &
 \end{array} \tag{1.E}$$

where the arm $[\alpha_{i_1}, \dots, \alpha_{i_{m_i}}]$ corresponds to $i \in \{1, \dots, n\}$ with $a_i \neq 0$, and the α_{ij} are given by the Hirzebruch–Jung continued fractions in (1.D). Furthermore, denoting $v = \#\{i \mid a_i \neq 0\}$ for the number of arms, we have $\beta = a + v$.

We first establish in 3.9 that π is the minimal resolution if and only if $\vec{x} \notin [0, \vec{c}]$. Theorem 1.2 is then proved by splitting into the two cases $\vec{x} \notin [0, \vec{c}]$ and $\vec{x} \in [0, \vec{c}]$, with the verification in both cases being rather different. Note that the case $\vec{x} \in [0, \vec{c}]$ is degenerate as $[0, \vec{c}]$ is a finite interval, containing only those \vec{x} of the form $a_i \vec{x}_i$ for some i and some $0 \leq a_i < p_i$. In this paper we are mostly interested in special CM modules and these are defined using the minimal resolution: this is why below the condition $\vec{x} \notin [0, \vec{c}]$ often appears.

We remark that for $0 \neq \vec{x} \in \mathbb{L}_+$, $S^{\vec{x}}$ is rarely a quotient singularity, and it is even more rare for it to be ADE. Nevertheless, the dual graphs of all quotient singularities k^2/G (where G is a small subgroup of $\mathrm{GL}(2, k)$) are known [B3], and so whether $S^{\vec{x}}$ is a quotient singularity can, if needed, be immediately determined by 1.2, after contracting (-1) -curves if necessary.

One key observation in this paper is that the construction of a minimal resolution $Y^{\vec{x}} \rightarrow \mathrm{Spec} S^{\vec{x}}$ allows us not only to construct many rational surface singularities with prescribed dual graphs (by taking suitable Veronese subrings of S), but furthermore we can use the stack structure to determine the special CM $S^{\vec{x}}$ -modules. Throughout, we denote by $\mathrm{SCM} S^{\vec{x}}$ the category of special CM $S^{\vec{x}}$ -modules; for definitions see §2.3.

As notation, recall that the i -series associated to the Hirzebruch–Jung continued fraction expansion $\frac{r}{a} = [\alpha_1, \dots, \alpha_m]$ is defined as $i_0 = r$, $i_1 = a$ and $i_t = \alpha_{t-1} i_{t-1} - i_{t-2}$ for all t with $2 \leq t \leq m+1$, and we denote

$$I(r, a) := \{i_0, i_1, \dots, i_{m+1}\}.$$

As convention $I(r, r) = \emptyset$. Also, for $\vec{y} \in \mathbb{L}$, we write $S(\vec{y})^{\vec{x}} := \bigoplus_{i \in \mathbb{Z}} S_{\vec{y}+i\vec{x}}$.

Theorem 1.3 (=3.10, 4.17). *If $\vec{x} \in \mathbb{L}_+$ with $\vec{x} \notin [0, \vec{c}]$, write \vec{x} in normal form $\vec{x} = \sum_{i=1}^n a_i \vec{x}_i + a\vec{c}$. Then*

$$\mathrm{SCM} S^{\vec{x}} = \mathrm{add}\{S(u\vec{x}_j)^{\vec{x}} \mid j \in [1, n], u \in I(p_j, p_j - a_j)\}.$$

This allows us to construct both $R = S^{\vec{x}}$, and its special CM modules, for (almost) every star shaped dual graph. We remark that this is the first time that special CM modules have been classified in any example with infinite CM representation type, and indeed, due to the non-tautness of the dual graph, in an uncountable family of examples. For simplicity in this paper, we restrict the explicitness to certain families of examples, and refer the reader to §5.2 for more details.

By construction, all the special CM $S^{\vec{x}}$ -modules have a natural \mathbb{Z} -grading, and we let N denote their sum. By definition the *reconstruction algebra* is defined to be $\Gamma_{\vec{x}} := \mathrm{End}_{S^{\vec{x}}}(N)$, and in this setting it inherits a \mathbb{N} -grading from the grading of the special CM modules in 1.3. In general, it is not generated in degree one over its degree zero piece, but nevertheless the degree zero piece is always some canonical algebra of Ringel. We state the first half of the following result vaguely, giving a much more precise description of the parameters in 4.23.

Proposition 1.4. *Suppose that $x \in \mathbb{L}_+$ with $\vec{x} \notin [0, \vec{c}]$.*

- (1) (=4.23) *The degree zero part of $\Gamma_{\vec{x}}$ is isomorphic to the canonical algebra $\Lambda_{\mathbf{q}, \mu}$, for some suitable parameters (\mathbf{q}, μ) .*
- (2) (=5.8) *For $\vec{s} := \sum_{i=1}^n \vec{x}_i$, then $\Gamma_{\vec{s}}$ is generated in degree one over its degree zero piece. Moreover the degree zero piece is the canonical algebra $\Lambda_{\mathbf{p}, \lambda}$.*

1.3. Geigle–Lenzing Weighted Projective Lines via Rational Surface Singularities. Motivated by the above, and also the fact that when studying curves it should not matter how we embed them into surfaces (and thus be independent of any self-intersection numbers that appear), we then investigate when $\mathrm{qgr}^{\mathbb{Z}} S^{\vec{x}} \simeq \mathrm{coh} \mathbb{X}$.

In very special cases, $\mathrm{coh} \mathbb{X}_{\mathbf{p}, \lambda}$ is already known to be equivalent to $\mathrm{qgr}^{\mathbb{Z}} R$ for some connected graded commutative ring R [GL2, 8.4]. The nicest situation is when the star-shaped dual graph is of Dynkin type, and further R is the ADE quotient singularity

associated to the Dynkin diagram via the McKay correspondence (with a slightly non-standard grading). However, all the previous attempts to link the weighted projective line to rational singularities have taken all self-intersection numbers to be -2 , which is well-known to restrict the possible configurations to ADE Dynkin type.

One of our main results is the following, which does not even require that $\vec{x} \in \mathbb{L}_+$.

Theorem 1.5 (=4.8). *Suppose that $\vec{x} = \sum_{i=1}^n a_i \vec{x}_i + a\vec{c}$ is not torsion, and denote $R := S^{\vec{x}}$. Then the following conditions are equivalent.*

- (1) *The natural functor $(-)^{\vec{x}}: \mathbf{CM}^{\mathbb{L}} S \rightarrow \mathbf{CM}^{\mathbb{Z}} R$ is an equivalence.*
- (2) *The natural functor $(-)^{\vec{x}}: \mathbf{qgr}^{\mathbb{L}} S \rightarrow \mathbf{qgr}^{\mathbb{Z}} R$ is an equivalence.*
- (3) *For any $\vec{z} \in \mathbb{L}$, the ideal $I^{\vec{z}} := S(\vec{z})^{\vec{x}} \cdot S(-\vec{z})^{\vec{x}}$ of R satisfies $\dim_k(R/I^{\vec{z}}) < \infty$.*
- (4) *$(p_i, a_i) = 1$ for all $1 \leq i \leq n$.*

Thus for a non-torsion element $\vec{x} = \sum_{i=1}^n a_i \vec{x}_i + a\vec{c}$ of \mathbb{L} , there is an equivalence $\mathbf{qgr}^{\mathbb{Z}} S_{\mathbf{p}, \lambda}^{\vec{x}} \simeq \mathbf{coh} \mathbb{X}_{\mathbf{p}, \lambda}$ if and only if $(p_i, a_i) = 1$ for all i with $1 \leq i \leq n$. Thus, by choosing a suitable \vec{x} , the weighted projective line can be defined using only connected \mathbb{N} -graded rational surface singularities. Also, we remark that in the case $(p_i, a_i) \neq 1$ we still have that $\mathbf{qgr} S^{\vec{x}}$ is equivalent to some weighted projective line, but the parameters are no longer (\mathbf{p}, λ) . We leave the details to §4.

Combining the above gives our next main result.

Corollary 1.6 (=4.8, 4.25). *Let $\vec{x} \in \mathbb{L}_+$ with $\vec{x} \notin [0, \vec{c}]$, and write $\vec{x} = \sum_{i=1}^n a_i \vec{x}_i + a\vec{c}$ in normal form. If $(p_i, a_i) = 1$ for all $1 \leq i \leq n$, then*

$$\mathbf{coh} \mathbb{X}_{\mathbf{p}, \lambda} \simeq \mathbf{qgr}^{\mathbb{Z}} S^{\vec{x}} \simeq \mathbf{qgr}^{\mathbb{Z}} \Gamma_{\vec{x}},$$

and further $\Gamma_{\vec{x}}$ is an \mathbb{N} -graded ring, with zeroth piece a canonical algebra.

In the case when $(p_i, a_i) \neq 1$ we have a similar result but again there is a change of parameters, so we refer the reader to 4.25 for details. Combining 1.6 with 1.4(2), we can view the weighted projective line $\mathbb{X}_{\mathbf{p}, \lambda}$ as an Artin–Zhang noncommutative projective scheme over the canonical algebra $\Lambda_{\mathbf{p}, \lambda}$ [M].

1.4. Some Particular Veronese Subrings. We then investigate in detail the particular Veronese subrings $S^{\vec{s}_a}$ for $\vec{s}_a := \vec{s} + a\vec{c}$ for some $a \geq 0$, and the special case $\vec{s} := \sum_{i=1}^n \vec{x}_i$. We call $S^{\vec{s}_a}$ the a -Wahl Veronese subring, and in this case, the singularities in (1.B) are all of the form $\frac{1}{p_i}(1, -1)$, which are cyclic Gorenstein and so have a resolution consisting of only (-2) -curves. Thus resolving the singularities in (1.B), by 1.2 we see that the dual graph of the minimal resolution of $\text{Spec } S^{\vec{s}_a}$ is

$$\begin{array}{c}
 \begin{array}{ccccccc}
 \bullet & -2 & & \bullet & -2 & & \bullet & -2 & & \bullet & -2 & & \bullet & -2 \\
 \vdots & & & \vdots & & & \vdots & & & \vdots & & & \vdots & \\
 \bullet & -2 & & \bullet & -2 & & \bullet & -2 & & \bullet & -2 & & \bullet & -2 \\
 \vdots & & & \vdots & & & \vdots & & & \vdots & & & \vdots & \\
 \bullet & -2 & & \bullet & -2 & & \bullet & -2 & & \bullet & -2 & & \bullet & -2 \\
 \vdots & & & \vdots & & & \vdots & & & \vdots & & & \vdots & \\
 \bullet & -2 & & \bullet & -2 & & \bullet & -2 & & \bullet & -2 & & \bullet & -2
 \end{array} \\
 \begin{array}{c}
 p_1-1 \quad p_2-1 \quad p_3-1 \quad \dots \quad p_n-1
 \end{array}
 \end{array}
 \quad (1.F)$$

$\begin{array}{c} \bullet \\ -n-a \end{array}$

where there are n arms, and the number of vertices on arm i is $p_i - 1$. It turns out that these particular Veronese subrings have many nice properties; not least by 1.1(4) they are precisely the Veronese subrings for which

$$\mathbf{D}^b(\mathbf{coh} Y^{\vec{x}}) \hookrightarrow \mathbf{D}^b(\mathbf{coh} \mathbb{T}^{\vec{x}})$$

is an equivalence. In §6 we investigate $S^{\vec{s}_a}$ in the case when (p_1, p_2, p_3) forms a Dynkin triple, in which case $S^{\vec{s}_a}$ is isomorphic to a quotient singularity by some finite subgroup of $\text{GL}(2, k)$ of type \mathbb{D} , \mathbb{T} , \mathbb{O} or \mathbb{I} (see 6.1 for details). In this situation $S^{\vec{s}_a}$ and its reconstruction algebra have a very nice relationship to the preprojective algebra of the

In the last section of the paper, finally we can explain the coincidence of the two motivating pictures, as a consequence of the following result.

Theorem 1.11 (=6.3). *Let R be the $(m-3)$ -Wahl Veronese subring associated with $(p_1, p_2, p_3) = (2, 3, 3)$, $(2, 3, 4)$ or $(2, 3, 5)$ and $m \geq 3$, and \mathfrak{R} its completion. Let $G \leq \mathbb{L}$ be the cyclic group generated by $(h(m-2)+1)\vec{\omega}$, where $h = 6, 12$ or 30 respectively. Then*

- (1) *There are equivalences $\text{vect } \mathbb{X} \simeq \text{CM}^{\mathbb{Z}} R$ and*

$$F: (\text{vect } \mathbb{X})/G \xrightarrow{\sim} \text{CM } \mathfrak{R}.$$

- (2) *For the canonical tilting bundle \mathcal{E} on \mathbb{X} , we have $\text{SCM } \mathfrak{R} = \text{add } F\mathcal{E}$.*

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Conventions. Throughout, k denotes an algebraically closed field of characteristic zero. All modules will be right modules, and for a ring A write $\text{mod } A$ for the category of finitely generated right A -modules. If G is an abelian group and A is a noetherian G -graded ring, $\text{gr}^G A$ will denote the category of finitely generated G -graded right A -modules. Throughout when composing maps fg will mean f then g , similarly for arrows ab will mean a then b . Note that with this convention $\text{Hom}_R(M, N)$ is an $\text{End}_R(M)^{\text{op}}$ -module and an $\text{End}_R(N)$ -module. For $M \in \text{mod } A$ we denote $\text{add } M$ to be the full subcategory consisting of summands of finite direct sums of copies of M .

2. PRELIMINARIES

2.1. Notation. We first fix notation. For $n \geq 0$, choose positive integers p_1, \dots, p_n with all $p_i \geq 2$, and set $\mathbf{p} := (p_1, \dots, p_n)$. Likewise, for pairwise distinct points $\lambda_1, \dots, \lambda_n \in \mathbb{P}^1$, and set $\boldsymbol{\lambda} := (\lambda_1, \dots, \lambda_n)$. Let $\ell_i(t_0, t_1) \in k[t_0, t_1]$ be the linear form defining λ_i .

Notation 2.1. To this data we associate

- (1) The abelian group $\mathbb{L} = \mathbb{L}(p_1, \dots, p_n)$ generated by the elements $\vec{x}_1, \dots, \vec{x}_n$ subject to the relations $p_1\vec{x}_1 = p_2\vec{x}_2 = \dots = p_n\vec{x}_n =: \vec{c}$. Note that \mathbb{L} is an ordered group with positive elements $\mathbb{L}_+ = \sum_{i=1}^n \mathbb{Z}_{\geq 0}\vec{x}_i$. Since $\mathbb{L}/\mathbb{Z}\vec{c} \cong \prod_{i=1}^n \mathbb{Z}/p_i\mathbb{Z}$ canonically, each $\vec{x} \in \mathbb{L}$ can be written uniquely in *normal form* as $\vec{x} = \sum_{i=1}^n a_i\vec{x}_i + a\vec{c}$ with $0 \leq a_i < p_i$ and $a \in \mathbb{Z}$. Then \vec{x} belongs to \mathbb{L}_+ if and only if $a \geq 0$.
- (2) The commutative k -algebra $S_{\mathbf{p}, \boldsymbol{\lambda}}$ defined as

$$S_{\mathbf{p}, \boldsymbol{\lambda}} := \frac{k[t_0, t_1, x_1, \dots, x_n]}{(x_i^{p_i} - \ell_i(t_0, t_1) \mid 1 \leq i \leq n)}.$$

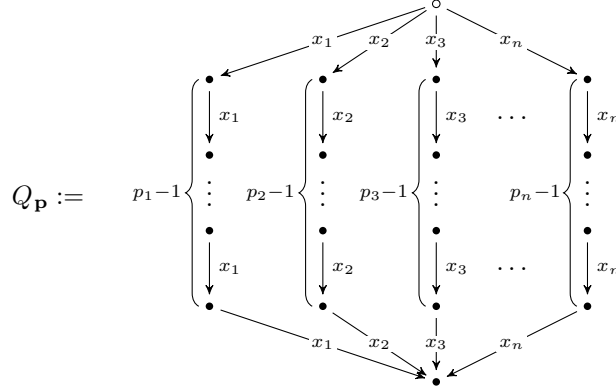
As in the introduction, this is \mathbb{L} -graded by $\deg x_i := \vec{x}_i$, and defines the weighted projective line $\mathbb{X}_{\mathbf{p}, \boldsymbol{\lambda}} := [(\text{Spec } S \setminus 0)/\text{Spec } k\mathbb{L}]$ and its coarse moduli space $X_{\mathbf{p}, \boldsymbol{\lambda}} := (\text{Spec } S \setminus 0)/\text{Spec } k\mathbb{L}$. The open cover $\text{Spec } S \setminus 0 = U_0 \cup U_1$ with $U_i := \text{Spec } S_{t_i}$ induces an open cover $X_{\mathbf{p}, \boldsymbol{\lambda}} = X_0 \cup X_1$ with $X_i := \text{Spec } (S_{t_i})_0$, where $(S_{t_i})_0$ is the degree zero part of S_{t_i} . Since by inspection $(S_{t_i})_0 = k[t_{1-i}/t_i]$, it follows that $X_{\mathbf{p}, \boldsymbol{\lambda}} \cong \mathbb{P}^1$.

When $n \geq 2$, often we choose $\lambda_1 = (1 : 0)$ and $\lambda_2 = (0 : 1)$, in which case $\ell_1 = t_1$, $\ell_2 = t_0$ and $\ell_i = \lambda_i t_0 - t_1$ for $3 \leq i \leq n$, and there is a presentation

$$S_{\mathbf{p}, \boldsymbol{\lambda}} = \frac{k[x_1, \dots, x_n]}{(x_i^{p_i} + x_1^{p_1} - \lambda_i x_2^{p_2} \mid 3 \leq i \leq n)}.$$

Moreover, when $n \geq 3$, we can further associate

(3) The quiver



(where there are n arms, and the number of vertices on arm i is $p_i - 1$).

(4) The *canonical algebra* $\Lambda_{\mathbf{p}, \lambda}$, namely the path algebra of the quiver $Q_{\mathbf{p}}$ subject to the relations

$$I := \langle x_1^{p_1} - \lambda_i x_2^{p_2} + x_i^{p_i} \mid 3 \leq i \leq n \rangle.$$

There is a degenerate definition of the canonical algebra if $1 \leq n \leq 2$; see [GL1].

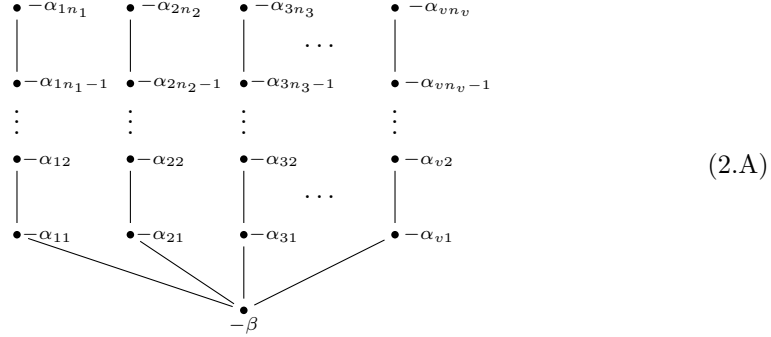
(5) The commutative k -algebra $R_{\mathbf{p}, \lambda}$, generated by u_1, \dots, u_n, v subject to the relations given by the 2×2 minors of the matrix

$$\begin{pmatrix} u_2 & u_3 & \dots & u_n & v^{p_2} \\ v^{p_1} & \lambda_3 u_3 + v^{p_3} & \dots & \lambda_n u_n + v^{p_n} & u_1 \end{pmatrix}$$

This is a connected \mathbb{N} -graded ring graded by $\deg u_1 := p_2$, $\deg u_2 := p_1$, $\deg v := 1$, and $\deg u_i := p_i$ for all $3 \leq i \leq n$.

We will also consider

(6) Star-shaped graphs of the form



where there are $v \geq 2$ arms, each $n_i \geq 1$, each $\alpha_{ij} \geq 2$ and $\beta \geq 1$. Later, we will assume $\beta \geq v$.

Let us briefly recall here some basic properties of $S = S_{\mathbf{p}, \lambda}$ from [GL1] (see also [HIMO, 3.15]) that we will require later. It is elementary that S is an \mathbb{L} -domain, i.e. a product of non-zero homogeneous elements is again non-zero. Recall that an element $x \in S$ is called an \mathbb{L} -prime if $S/(x)$ is an \mathbb{L} -domain.

Proposition 2.2. *S is \mathbb{L} -factorial, i.e. every non-zero homogeneous element in S is a product of \mathbb{L} -prime elements in S .*

2.2. Preliminaries on Rational Surface Singularities. We briefly review some combinatorics associated to rational surface singularities. Recall that a commutative k -algebra R is called a *rational surface singularity* if $\dim R = 2$ and there exists a resolution $f: X \rightarrow \operatorname{Spec} R$ such that $\mathbf{R}f_* \mathcal{O}_X = \mathcal{O}_R$. If this property holds for one resolution, it holds for all resolutions [KM, 5.10].

In our setting later R will be a rational surface singularity with a unique singular point, at the origin. Completing at this maximal ideal to give \mathfrak{R} , in the minimal resolution

$Y \rightarrow \text{Spec } \mathfrak{R}$ the fibre above the origin is well-known to be a tree (i.e. a finite connected graph with no cycles) of \mathbb{P}^1 s denoted $\{E_i\}_{i \in I}$. Their self-intersection numbers satisfy $E_i \cdot E_i \leq -2$, and moreover the intersection matrix $(E_i \cdot E_j)_{i,j \in I}$ is negative definite. We encode the intersection matrix in the form of the labelled dual graph:

Definition 2.3. Suppose that $\{E_i\}_{i \in I}$ is a collection of \mathbb{P}^1 s forming the exceptional locus in a resolution of some affine rational surface singularity. The dual graph is defined as follows: for each curve draw a vertex, and join two vertices if and only if the corresponding curves intersect. Furthermore label every vertex with the self-intersection number of the corresponding curve.

Thus, given a complete local rational surface singularity, we obtain a labelled tree. Conversely, suppose that T is a tree, with vertices denoted E_1, \dots, E_n , labelled with integers w_1, \dots, w_n . To this data we associate the symmetric matrix $M_T = (b_{ij})_{1 \leq i, j \leq n}$ with b_{ii} defined by $b_{ii} := w_i$, and b_{ij} (with $i \neq j$) defined to be the number of edges linking the vertices E_i and E_j . We denote the free abelian group generated by the vertices E_i by \mathcal{Z} , and call its elements *cycles*. The matrix M_T defines a symmetric bilinear form $(-, -)$ on \mathcal{Z} and in analogy with the geometry, we will often write $Y \cdot Z$ instead of (Y, Z) , and consider

$$\mathcal{Z}_{\text{top}} := \{Z = \sum_{i=1}^n a_i E_i \in \mathcal{Z} \mid Z \neq 0, \text{ all } a_i \geq 0, \text{ and } Z \cdot E_i \leq 0 \text{ for all } i\}.$$

If there exists $Z \in \mathcal{Z}_{\text{top}}$ such that $Z \cdot Z < 0$, then automatically M_T is negative definite [A, Prop 2(ii)]. In this case, \mathcal{Z}_{top} admits a unique smallest element Z_f , called the *fundamental cycle*. Whenever all the coefficients in Z_f are one, the fundamental cycle is said to be *reduced*.

We now consider the case of the labelled graph (2.A) and calculate some combinatorics that will be needed later. Denoting the set of vertices of (2.A) by I , considering $Z := \sum_{i \in I} E_i$ it is easy to see that

$$(-Z \cdot E_i)_{i \in I} = \begin{array}{ccccccc} & \alpha_{1n_1}-1 & \alpha_{2n_2}-1 & \alpha_{3n_3}-1 & \dots & \alpha_{vn_v}-1 & \\ & | & | & | & & | & \\ \alpha_{1n_1}-1-2 & \alpha_{2n_2}-1-2 & \alpha_{3n_3}-1-2 & & & \alpha_{vn_v}-1-2 & \\ \vdots & \vdots & \vdots & & & \vdots & \\ \alpha_{12}-2 & \alpha_{22}-2 & \alpha_{32}-2 & \dots & & \alpha_{v2}-2 & \\ | & | & | & & & | & \\ \alpha_{11}-2 & \alpha_{21}-2 & \alpha_{31}-2 & \dots & & \alpha_{v1}-2 & \\ & & & & & & \\ & & & & & & \beta-v \end{array} \quad (2.B)$$

and so Z satisfies $Z \cdot E_i \leq 0$ for all $i \in I$ if and only if $\beta \geq v$. Since \mathcal{Z}_{top} does not contain elements smaller than Z , the fundamental cycle Z_f is given by $Z = \sum_{i \in I} E_i$ if and only if $\beta \geq v$. In this case Z_f is reduced.

We remark that in general there will be many singularities with dual graph (2.A), and indeed a labelled graph T is called *taut* if there exists a unique (up to isomorphism in the category of complete local k -algebras) rational surface singularity which has T for its dual graph of its minimal resolution. It is well known that the labelled graph (2.A) is taut if and only if $n = 3$ [L].

2.3. Preliminaries on Reconstruction Algebras. Let R be a rational surface singularity. A CM R -module M is called *special* if $\text{Ext}_R^1(M, R) = 0$ [IW], and we write $\text{SCM } R$ for the category of special CM R -modules.

The following local-to-global lemma is useful. In particular, if R has a unique singular point \mathfrak{m} , to conclude that $\text{add } M = \text{SCM } R$ it suffices to check this complete locally at \mathfrak{m} .

Lemma 2.4. Let R be a rational surface singularity, and $M \in \text{CM } R$. If $\text{add } \widehat{M}_{\mathfrak{m}} = \text{SCM } \widehat{R}_{\mathfrak{m}}$ for all $\mathfrak{m} \in \text{Max } R$, then $\text{add } M = \text{SCM } R$.

Proof. Since Ext groups localise and complete, certainly $M \in \text{SCM } R$ and thus $\text{add } M \subseteq \text{SCM } R$. Next, let $X \in \text{SCM } R$. Then $\text{add } \widehat{X}_{\mathfrak{m}} \subseteq \text{SCM } \widehat{R}_{\mathfrak{m}}$ for all $\mathfrak{m} \in \text{Max } R$, so by assumption $\text{add } \widehat{X}_{\mathfrak{m}} \subseteq \text{add } \widehat{M}_{\mathfrak{m}}$ for all $\mathfrak{m} \in \text{Max } R$. By [IW2, 2.26] we conclude that $\text{add } X \subseteq \text{add } M$, so $X \in \text{add } M$ and thus $\text{add } M \supseteq \text{SCM } R$ also holds. \square

The following asserts that a global additive generator of $\text{SCM } R$ exists, regardless of the number of points in the singular locus.

Theorem 2.5 ($=[\text{VdB}]$). *Let R be a rational surface singularity, and $\pi: X \rightarrow \text{Spec } R$ the minimal resolution. Then*

- (1) *There exists $M \in \text{SCM } R$ such that $\text{SCM } R = \text{add } M$.*
- (2) *There is a triangle equivalence $\text{D}^b(\text{mod } \text{End}_R(M)) \cong \text{D}^b(\text{coh } X)$.*

Proof. This is known but usually only stated when R is complete, so for the convenience of the reader we provide a proof. By $[\text{VdB}, 3.2.5]$ there is a progenerator $\mathcal{O}_X \oplus \mathcal{M}$ for the category of perverse sheaves (with perversity -1), which induces an equivalence

$$\text{D}^b(\text{mod } \text{End}_X(\mathcal{O}_X \oplus \mathcal{M})) \cong \text{D}^b(\text{coh } X).$$

There is an isomorphism $\text{End}_X(\mathcal{O}_X \oplus \mathcal{M}) \cong \text{End}_R(R \oplus \pi_* \mathcal{M})$ by $[\text{DW}, 4.1]$. Furthermore, $\mathcal{O}_X \oplus \mathcal{M}$ remains a progenerator under flat base change $[\text{VdB}, 3.1.6]$, so $\text{add } \widehat{M}_{\mathfrak{m}} = \text{SCM } \widehat{R}_{\mathfrak{m}}$ by $[\text{W6}, \text{IW}]$. The result then follows using 2.4. \square

Definition 2.6. *For any $M \in \text{SCM } R$ such that $\text{SCM } R = \text{add } M$, we call $\text{End}_R(M)$ the reconstruction algebra.*

In this global setting, the reconstruction algebra is only defined up to Morita equivalence. Only after completing R , or in certain other settings (see 2.8), will there be a canonical choice.

When \mathfrak{R} is a complete local rational surface singularity with minimal resolution $X \rightarrow \text{Spec } \mathfrak{R}$, there is a much more explicit description of the additive generator of $\text{SCM } \mathfrak{R}$. Let $\{E_i \mid i \in I\}$ denote the exceptional curves, then for each $i \in I$, by $[\text{W6}]$ there exists a CM \mathfrak{R} -module M_i such that $H^1(\mathcal{M}_i^\vee) = 0$ and $c_1(\mathcal{M}_i) \cdot E_j = \delta_{ij}$ hold, where $\mathcal{M}_i := \pi^*(M_i)^{\vee\vee}$ for $(-)^{\vee} = \mathcal{H}om_X(-, \mathcal{O}_X)$. It is shown in $[\text{W6}, 1.2]$ that $\{M_i \mid i \in I\}$ are precisely the indecomposable non-free objects in $\text{SCM } \mathfrak{R}$, hence $\mathfrak{R} \oplus \bigoplus_{i \in I} M_i$ is the natural additive generator for $\text{SCM } \mathfrak{R}$.

Definition 2.7. *Let \mathfrak{R} be a complete local rational surface singularity. We call $\Gamma := \text{End}_{\mathfrak{R}}(\mathfrak{R} \oplus (\bigoplus_{i \in I} M_i))$ the reconstruction algebra of \mathfrak{R} .*

Remark 2.8. If R is a rational surface singularity with a unique singular point, and there exist $L_i \in \text{CM } R$ such that $\widehat{L}_i \cong M_i$ for all i , then we also use the letter Γ to denote the particular reconstruction algebra

$$\Gamma := \text{End}_R(R \oplus \bigoplus_{i \in I} L_i)$$

of R . Such L_i are not guaranteed to exist, in general.

In the complete local setting, the quiver of Γ , and the number of its relations, is completely determined by the intersection theory.

Theorem 2.9 ($=[\text{W2}, 3.3]$). *Let \mathfrak{R} be a complete local rational surface singularity. The quiver and the numbers of relations of Γ is given as follows: for every $i \in I$ associate a vertex labelled i corresponding to M_i , and also associate a vertex labelled \circ corresponding to \mathfrak{R} . Then the number of arrows and relations between the vertices is*

	Number of arrows	Number of relations
$i \rightarrow j$	$(E_i \cdot E_j)_+$	$(-1 - E_i \cdot E_j)_+$
$\circ \rightarrow \circ$	0	$-Z_K \cdot Z_f + 1 = -1 - Z_f \cdot Z_f$
$i \rightarrow \circ$	$-E_i \cdot Z_f$	0
$\circ \rightarrow i$	$((Z_K - Z_f) \cdot E_i)_+$	$((Z_K - Z_f) \cdot E_i)_-$

where for $a \in \mathbb{Z}$

$$a_+ := \begin{cases} a & \text{if } a \geq 0 \\ 0 & \text{if } a < 0 \end{cases} \quad \text{and} \quad a_- = \begin{cases} 0 & \text{if } a \geq 0 \\ -a & \text{if } a < 0 \end{cases},$$

and the canonical cycle Z_K is by definition the rational cycle defined by the condition $Z_K \cdot E_i = E_i^2 + 2$ for all $i \in I$.

2.4. Hirzebruch–Jung Continued Fraction Combinatorics. We review briefly the notation and combinatorics surrounding dimension two cyclic quotient singularities.

Definition 2.10. For $r, a \in \mathbb{N}$ with $r > a$ the group $G = \frac{1}{r}(1, a)$ is defined by

$$G = \left\langle \zeta := \begin{pmatrix} \varepsilon & 0 \\ 0 & \varepsilon^a \end{pmatrix} \right\rangle \leq \mathrm{GL}(2, k),$$

where ε is a primitive r^{th} root of unity. By abuse of notation, we also denote the corresponding quotient singularity S^G for $S = k[x, y]$ by $\frac{1}{r}(1, a)$.

Remark 2.11. In the literature it is often assumed that the greatest common divisor (r, a) is 1, which is equivalent to the group having no pseudoreflections. However we do not make this assumption, since in our construction later singularities with pseudoreflections naturally appear.

Provided that $a \neq 0$, we consider the Hirzebruch–Jung continued fraction expansion of $\frac{r}{a}$, namely

$$\frac{r}{a} = \alpha_1 - \frac{1}{\alpha_2 - \frac{1}{\alpha_3 - \frac{1}{\ddots}}}} := [\alpha_1, \dots, \alpha_n]$$

with each $\alpha_i \geq 2$. The labelled Dynkin diagram

$$\begin{array}{ccccccc} \bullet & \text{---} & \bullet & \text{---} & \cdots & \text{---} & \bullet & \text{---} & \bullet \\ -\alpha_1 & & -\alpha_2 & & & & -\alpha_{n-1} & & -\alpha_n \end{array}$$

is precisely the dual graph of the minimal resolution of $k^2/\frac{1}{r}(1, a)$ [R1, Satz8]. Note that [R1] assumed the condition $(r, a) = 1$, but the result holds generally: if we write $h := (r, a)$, then the quotient singularities $\frac{1}{r}(1, a)$ and $\frac{1}{r/h}(1, a/h)$ are isomorphic, and furthermore both have the same Hirzebruch–Jung continued fraction expansion.

Definition 2.12. For integers $1 \leq a < r$ as above, denote the continued fraction expansion $\frac{r}{a} = [\alpha_1, \dots, \alpha_n]$. We associate combinatorial series defined as follows:

- (1) The i -series is defined as $i_0 = r$, $i_1 = a$ and

$$i_t = \alpha_{t-1}i_{t-1} - i_{t-2}$$

for all t with $2 \leq t \leq n+1$.

- (2) Denote the i -series of $\frac{r}{r-a}$ by i_t , then the j -series is defined $j_t = i_{n+1-t}$ for all t with $0 \leq t \leq n+1$.

As in the introduction, we denote $I(r, a) := \{i_0, i_1, \dots, i_{n+1}\}$, where by convention $I(r, r) = \emptyset$. The following lemma is elementary, and will be needed later.

Lemma 2.13. For integers $1 \leq a < r$, $I(r, a) = [0, r]$ if and only if $a = r - 1$.

For cyclic quotient singularities $G = \frac{1}{r}(1, a)$, consider

$$S_t := \{f \in k[x, y] \mid \sigma \cdot f = \varepsilon^t f\},$$

for $t \in [0, r]$, and note that $S_0 \cong S_r$. Further, for k with $0 \leq k \leq r-1$, we say that a monomial $x^m y^n$ has weight k if $m + an = k \pmod{r}$, that is $x^m y^n \in S_k$. It is the i -series that determines which CM S^G -modules are special.

Theorem 2.14. For $G = \frac{1}{r}(1, a)$,

- (1) [H2] $\mathrm{CM} S^G = \mathrm{add}\{S_t \mid t \in [0, r]\}$.
(2) [W5] $\mathrm{SCM} S^G = \mathrm{add}\{S_t \mid t \in I(r, a)\}$.

Proof. Both results are usually stated in the complete case, with no pseudoreflections, so since we are working more generally, we give the proof. Since S^G has a unique singular point, by 2.4 (and its counterpart in the $\mathrm{CM} S^G$ case) it suffices to prove both results in the complete local setting. In this case, when $(r, a) = 1$, part (1) is [H2] and part (2) is [W5]. When $(r, a) \neq 1$, the result is still true since $\frac{1}{r}(1, a) = \frac{1}{r/h}(1, a/h)$ for $h := (r, a)$. \square

In what follows, we will require a different characterization of members of the i -series, by reinterpreting of a result of Ito [I, 3.7]. As notation, if $(r, a) = 1$ then the G -basis $B(G)$ is defined to be the set of monomials which are not divisible by any G -invariant monomial. We usually draw $B(G)$ in a 2×2 grid.

Example 2.15. Consider $G = \frac{1}{17}(1, 10)$. Then $B(G)$ is

1	y	y^2	y^3	y^4	y^5	...	y^{16}
x	xy	xy^2	xy^3	xy^4			
x^2	x^2y	x^2y^2	x^2y^3	x^2y^4			
x^3	x^3y	x^3y^2	x^3y^3	x^3y^4			
x^4	x^4y	x^4y^2					
x^5	x^5y	x^5y^2					
x^6	x^6y	x^6y^2					
x^7							
\vdots							
x^{16}							

For $G = \frac{1}{r}(1, a)$ with $(r, a) = 1$, recall that the L -space $L(G)$ is defined to be

$$L(G) := \{1, x, \dots, x^{r-1}, y, \dots, y^{r-1}\},$$

so called since in the 2×2 grid the shape of $L(G)$ looks like the letter L.

Theorem 2.16 (= [1, 3.7]). *When $(r, a) = 1$, the elements of $I(r, a)$ are precisely those numbers in $[0, r]$ that do not appear as weights of monomials in the region $B(G) \setminus L(G)$.*

Example 2.17. Consider $G = \frac{1}{17}(1, 10)$. Then $B(G) \setminus L(G)$ is the region

1	y	y^2	y^3	y^4	y^5	\dots	y^{16}
x	xy	xy^2	xy^3	xy^4			
x^2	x^2y	x^2y^2	x^2y^3	x^2y^4			
x^3	x^3y	x^3y^2	x^3y^3	x^3y^4			
x^4	x^4y	x^4y^2					
x^5	x^5y	x^5y^2					
x^6	x^6y	x^6y^2					
x^7							
\vdots							
x^{16}							

Replacing the monomials in the above region by their corresponding weights gives

11	4	14	7
12	5	15	8
13	6	16	9
14	7		
15	8		
16	9		

and so by 2.16, the i -series consists of those numbers that do not appear in the above region, which are precisely the numbers 0, 1, 2, 3, 10 and 17. Indeed, in this example $\frac{17}{10} = [2, 4, 2, 2]$ and the i -series is

$$i_0 = 17 > i_1 = 10 > i_2 = 3 > i_3 = 2 > i_4 = 1 > i_5 = 0.$$

The following lemma, which we use later, is an extension of 2.16. For integers $r > 0$ and k , write $[k]_r$ for the unique integer k' satisfying $0 \leq k' \leq r-1$ and $k - k' \in r\mathbb{Z}$.

Lemma 2.18. *Assume $(r, a) = 1$. For $0 \leq u \leq r-1$, the following are equivalent.*

- (1) $u \in I(r, r-a)$.
- (2) u does not appear in $B(G) \setminus L(G)$ for $G := \frac{1}{r}(1, -a)$.
- (3) For every integer $\ell \geq 1$, there exists an integer $m \in [1, \ell]$ such that $[u + \ell a - 1]_r \geq [ma - 1]_r$.

Proof. (1) \Leftrightarrow (2) This is 2.16.

(2) \Leftrightarrow (3) We will establish the following claim: u does not appear in column ℓ of $B(G) \setminus L(G)$ if and only if there exists an integer m satisfying $1 \leq m \leq \ell$ and $[u + \ell a - 1]_r \geq [ma - 1]_r$.

The first row of $B(G)$ is $0, -a, -2a, -3a, \dots$. Now for each m with $1 \leq m \leq \ell$, we find the first occurrence of weight 0 in column m , and use this to draw the following diagram:

$$\begin{array}{ccc}
 -ma & \cdots & -\ell a \\
 \hline
 1-ma & \cdots & 1-\ell a \\
 2-ma & \cdots & 2-\ell a \\
 \vdots & \ddots & \vdots \\
 -2 & \cdots & -2+(m-\ell)a \\
 -1 & \cdots & -1+(m-\ell)a \\
 \hline
 0 & \cdots & (m-\ell)a
 \end{array}$$

The column ℓ of $B(G) \setminus L(G)$ is the intersection, over all m with $1 \leq m \leq \ell$, of the above dotted regions. It is clear that u does not appear in the dotted region in the above diagram if and only if $[u + \ell a - 1]_r \geq [ma - 1]_r$. The claim follows. \square

Notation 2.19. Throughout the remainder of the paper, to aid readability we will use the following simplified notation.

Notation	Meaning	Simplified Notation
$\mathbb{X}_{\mathbf{p}, \lambda}$	Weighted projective line	\mathbb{X}
$S_{\mathbf{p}, \lambda}$	Defining ring of $\mathbb{X}_{\mathbf{p}, \lambda}$	S
$\Lambda_{\mathbf{p}, \lambda}$	Canonical algebra	Λ
$S_{\mathbf{p}, \lambda}^{\vec{x}}$	Veronese of $\mathbb{S}_{\mathbf{p}, \lambda}$ with respect to $\vec{x} \in \mathbb{L}$	$S^{\vec{x}}$
$Y_{\mathbf{p}, \lambda}^{\vec{x}}$	Resolution of $\text{Spec } S_{\mathbf{p}, \lambda}^{\vec{x}}$ in (1.C)	$Y^{\vec{x}}$

Throughout it will be implicit that we are working generally, with parameters (\mathbf{p}, λ) .

3. THE TOTAL SPACE \mathbb{T}

Throughout this section we work with arbitrary parameters (\mathbf{p}, λ) and use the simplified notation of 2.19. As in the introduction, we choose $0 \neq \vec{x} \in \mathbb{L}_+$, and consider the Veronese $S^{\vec{x}}$, and the total space stack

$$\mathbb{T}^{\vec{x}} = \text{Tot}(\mathcal{O}_{\mathbb{X}}(-\vec{x})) := [(\text{Spec } S \setminus 0 \times \text{Spec } k[t]) / \text{Spec } k\mathbb{L}],$$

where \mathbb{L} acts on t with weight $-\vec{x}$. There is a natural projection $q: \text{Tot}(\mathcal{O}_{\mathbb{X}}(-\vec{x})) \rightarrow \mathbb{X}$, and a natural map $g: \text{Tot}(\mathcal{O}_{\mathbb{X}}(-\vec{x})) \rightarrow T^{\vec{x}}$ to its coarse moduli space. There is also a map $f: \mathbb{X} \rightarrow X$ from \mathbb{X} to its coarse moduli space, and there is an obvious morphism $p: T^{\vec{x}} \rightarrow X$. Also, either simply by construction of $T^{\vec{x}}$, or by an easy Čech calculation, it follows that

$$H^0(\mathcal{O}_{T^{\vec{x}}}) = \bigoplus_{k \geq 0} H^0(\mathbb{X}, \mathcal{O}_{\mathbb{X}}(k\vec{x}))t^k \cong S^{\vec{x}}$$

and so there is a natural map $T^{\vec{x}} \rightarrow \text{Spec } S^{\vec{x}}$, which we denote by ψ . Furthermore, since $0 \neq \vec{x} \in \mathbb{L}_+$, necessarily $\dim S^{\vec{x}} = 2$ and by inspection the morphism ψ is projective birational.

We let $\varphi: Y^{\vec{x}} \rightarrow T^{\vec{x}}$ denote the minimal resolution of $T^{\vec{x}}$, and consider the composition $\pi: Y^{\vec{x}} \rightarrow T^{\vec{x}} \rightarrow \text{Spec } S^{\vec{x}}$. We remark that this composition need not be the minimal resolution of $\text{Spec } S^{\vec{x}}$, and indeed later we give a precise criterion for when it is. Nevertheless, as in the introduction, we summarize the above information in the following commutative diagram

$$\begin{array}{ccccc}
 \mathbb{T}^{\vec{x}} = \text{Tot}(\mathcal{O}_{\mathbb{X}}(-\vec{x})) & \xrightarrow{q} & \mathbb{X} & & \\
 \downarrow g & & \downarrow f & & \\
 Y^{\vec{x}} & \xrightarrow{\varphi} & T^{\vec{x}} & \xrightarrow{p} & X \cong \mathbb{P}^1 \\
 \searrow \pi & & \downarrow \psi & & \\
 & & \text{Spec } S^{\vec{x}} & &
 \end{array} \tag{3.A}$$

3.1. Tilting on \mathbb{T} and T . Write $\mathcal{V} := \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1) \in \text{coh } \mathbb{P}^1$, and $\mathcal{E} := \bigoplus_{\vec{y} \in [0, \vec{c}]} \mathcal{O}_{\mathbb{X}}(\vec{y}) \in \text{coh } \mathbb{X}$. The following result is well known.

Theorem 3.1. *The following statements hold.*

- (1) \mathcal{V} is a tilting bundle on \mathbb{P}^1 .
- (2) \mathcal{E} is a tilting bundle on \mathbb{X} .

Proof. Part (1) is [B1] and part (2) is [GL1]. \square

In this subsection we lift these tilting bundles to tilting bundles on both $T^{\vec{x}}$ and $\mathbb{T}^{\vec{x}}$, and from this we deduce that $S^{\vec{x}}$ is a rational surface singularity.

Theorem 3.2. *If $0 \neq \vec{x} \in \mathbb{L}_+$, then $q^*\mathcal{E}$ is a tilting bundle on $\mathbb{T}^{\vec{x}}$ such that*

$$\begin{array}{ccc} \text{D}^b(\text{Qcoh } \mathbb{T}^{\vec{x}}) & \xrightarrow[\sim]{\mathbf{R}\text{Hom}_{\mathbb{T}^{\vec{x}}}(q^*\mathcal{E}, -)} & \text{D}^b(\text{Mod End}_{\mathbb{T}^{\vec{x}}}(q^*\mathcal{E})) \\ \mathbf{R}q_* \downarrow & & \downarrow \text{res} \\ \text{D}^b(\text{Qcoh } \mathbb{X}) & \xrightarrow[\sim]{\mathbf{R}\text{Hom}_{\mathbb{X}}(\mathcal{E}, -)} & \text{D}^b(\text{Mod } \Lambda) \end{array}$$

commutes, where Λ is the canonical algebra.

Proof. To simplify, we drop all \vec{x} from the notation and set $\mathbb{T} := \text{Tot}(\mathcal{O}_{\mathbb{X}}(-\vec{x}))$. The generation argument is standard, as in [AU, 4.1] and [B2, 4.1], namely if $M \in \text{D}(\text{Qcoh } \mathbb{T})$ with $\text{Hom}_{\text{D}(\mathbb{T})}(q^*\mathcal{E}, M[i]) = 0$ for all i , then $\text{Hom}_{\text{D}(\mathbb{X})}(\mathcal{E}, q_*M[i]) = 0$ for all i , so since \mathcal{E} generates \mathbb{X} , $q_*M = 0$ and so since q is affine $M = 0$. Hence $q^*\mathcal{E}$ generates $\text{D}(\text{Qcoh } \mathbb{T})$, so since $\text{D}(\text{Qcoh } \mathbb{T})$ is compactly generated, $q^*\mathcal{E}$ is a classical generator of $\text{Perf}(\mathbb{T})$.

For Ext vanishing,

$$\begin{aligned} \text{Ext}_{\mathbb{T}}^1(q^*\mathcal{E}, q^*\mathcal{E}) &\cong \text{Ext}_{\mathbb{X}}^1(\mathcal{E}, q_*q^*\mathcal{E}) && \text{(by adjunction)} \\ &\cong \text{Ext}_{\mathbb{X}}^1(\mathcal{E}, \bigoplus_{k \geq 0} \mathcal{E} \otimes \mathcal{O}_{\mathbb{X}}(k\vec{x})) && \text{(by projection formula)} \\ &\cong \bigoplus_{k \geq 0} \text{Ext}_{\mathbb{X}}^1(\mathcal{E}, \mathcal{E} \otimes \mathcal{O}_{\mathbb{X}}(k\vec{x})) \\ &\cong \bigoplus_{k \geq 0} \bigoplus_{i \in [0, \vec{c}]} \bigoplus_{j \in [0, \vec{c}]} H^1(\mathbb{X}, \mathcal{O}_{\mathbb{X}}(i - j + k\vec{x})) \\ &\cong \bigoplus_{k \geq 0} \bigoplus_{i \in [0, \vec{c}]} \bigoplus_{j \in [0, \vec{c}]} H^0(\mathbb{X}, \mathcal{O}_{\mathbb{X}}(\vec{\omega} - i + j - k\vec{x}))^* && \text{(by Serre duality)} \end{aligned}$$

It is well known that $H^0(\mathbb{X}, \mathcal{O}_{\mathbb{X}}(\vec{y})) = S_{\vec{y}} = 0$ if $\vec{y} \notin \mathbb{L}_+$, so it suffices to check that $\vec{\omega} - i + j - k\vec{x} \notin \mathbb{L}_+$ for all $k \geq 0$ and all $i, j \in [0, \vec{c}]$. Since $0 \neq \vec{x} \in \mathbb{L}_+$, clearly it suffices to check $i = k = 0$ and $j = \vec{c}$, this being the most positive case. But $\vec{\omega} = (n-2)\vec{c} - \sum_{t=1}^n \vec{x}_t$, and so $\vec{\omega} + \vec{c} = (n-1)\vec{c} - \sum_{t=1}^n \vec{x}_t \notin \mathbb{L}_+$, as required. Since \mathbb{X} is hereditary, the above proof also shows that higher Exts also vanish.

The commutativity is just adjunction $\mathbf{R}\text{Hom}_{\mathbb{X}}(\mathcal{E}, \mathbf{R}q_*(-)) \cong \mathbf{R}\text{Hom}_{\mathbb{T}}(q^*\mathcal{E}, -)$. \square

Theorem 3.3. *If $0 \neq \vec{x} \in \mathbb{L}_+$, then $p^*\mathcal{V}$ is a tilting bundle on $T^{\vec{x}}$.*

Proof. As above, when possible we drop all \vec{x} from the notation. The generation argument is identical to the argument in 3.2. The Ext-vanishing is also similar, namely if we denote $\mathcal{F} := \mathcal{O}_{\mathbb{X}} \oplus \mathcal{O}_{\mathbb{X}}(\vec{c})$ then,

$$\begin{aligned} \text{Ext}_T^i(p^*\mathcal{V}, p^*\mathcal{V}) &\cong \text{Ext}_T^i(p^*\mathcal{V}, g_*\mathcal{O}_T \otimes_T p^*\mathcal{V}) && (g_*\mathcal{O}_T = \mathcal{O}_T) \\ &\cong \text{Ext}_T^i(p^*\mathcal{V}, g_*g^*p^*\mathcal{V}) && \text{(projection formula)} \\ &\cong \text{Ext}_{\mathbb{T}}^i(g^*p^*\mathcal{V}, g^*p^*\mathcal{V}) && \text{(adjunction)} \\ &\cong \text{Ext}_{\mathbb{T}}^i(q^*f^*\mathcal{V}, q^*f^*\mathcal{V}) && \text{(commutativity of (3.A))} \\ &\cong \text{Ext}_{\mathbb{T}}^i(q^*\mathcal{F}, q^*\mathcal{F}) \end{aligned}$$

which is zero by 3.2 since $q^*\mathcal{F}$ is a summand of $q^*\mathcal{E}$. \square

The following two results give precise information regarding the singularities in $T^{\vec{x}}$.

Proposition 3.4. $T^{\vec{x}}$ is a surface containing the coarse module $X \cong \mathbb{P}^1$ of \mathbb{X} . Moreover $T^{\vec{x}}$ is normal, and all its singularities are isolated and lie on X .

Proof. We use the presentation of S given in 2.1:

$$S = \frac{k[x_1, \dots, x_n]}{(x_i^{p_i} + x_1^{p_1} - \lambda_i x_2^{p_2} \mid 3 \leq i \leq n)}.$$

The open cover $\text{Spec } S \setminus 0 = U_1 \cup U_2$ with $U_i := \text{Spec } S_{x_i}$ induces an open cover $(\text{Spec } S \setminus 0) \times \text{Spec } k[t] = U'_1 \cup U'_2$ with $U'_i := U_i \times \text{Spec } k[t] = \text{Spec } S[t]_{x_i}$ for $i = 1, 2$. Thus $T^{\vec{x}}$ has an open cover $T^{\vec{x}} = V_1 \cup V_2$ where $V_i := \text{Spec } (S[t]_{x_i})_0$ and $(S[t]_{x_i})_0$ is the degree zero part of $S[t]_{x_i}$. As in 2.1(2), the curve $X_i := \text{Spec } (S_{x_i})_0$ in V_i for $i = 1, 2$ gives the coarse moduli $X = X_1 \cup X_2 \cong \mathbb{P}^1$ of \mathbb{X} .

Fix $i = 1, 2$ and let $A := S[t]_{x_i}$ and $B := (S[t]_{x_i})_0$ so that $V_i = \text{Spec } B$. We first claim that B is a normal domain. Since S is an \mathbb{L} -factorial \mathbb{L} -domain by 2.2, so are $S[t]$ and A . Thus the degree zero part B of A is a domain. To prove that B is normal, assume that an element x in the quotient field of B satisfies an equality $x^m + b_1 x^{m-1} + \dots + b_m = 0$ for some $b_i \in B$. Write $x = y/z$ for homogeneous elements y and z in A which do not have a common \mathbb{L} -prime factor. Since $y^m = -z(b_1 y^{m-1} + \dots + b_m z^{m-1})$ holds, z must be a unit in A . Thus we have $x \in A$ and $x \in B$, and the assertion follows.

Consider next the \mathbb{Z} -grading on $S[t]$ defined by $\deg t = 1$ and $\deg x = 0$ for any $x \in S$. This gives a \mathbb{Z} -grading on B such that $B = \bigoplus_{i \geq 0} B_i$ and $B_0 = (S_{x_i})_0$. Since B is a \mathbb{Z} -graded finitely generated k -algebra, by the Jacobian criterion, there is a \mathbb{Z} -graded ideal I of B such that $\text{Sing } B = \text{Spec}(B/I)$. Since B is a normal domain of dimension two, $\text{Sing } B$ consists of finitely many closed points and hence $\dim_k(B/I) < \infty$ holds. Since I is \mathbb{Z} -graded, it contains $\bigoplus_{i \geq \ell} B_i$ for $\ell \gg 0$ and hence \sqrt{I} contains $\bigoplus_{i > 0} B_i$. Consequently, $\text{Sing } B$ is contained in $\text{Spec } B_0 \subset X$. \square

Next we prove the following.

Proposition 3.5. On $X \cong \mathbb{P}^1$, complete locally the singularities of $T^{\vec{x}}$ are of the form

$$\begin{array}{ccccccc} \lambda_1 & \lambda_2 & \lambda_3 & & \mathbb{P}^1 & & \lambda_n \\ \frac{1}{p_1}(1, -a_1) & \frac{1}{p_2}(1, -a_2) & \frac{1}{p_3}(1, -a_3) & \dots & & & \frac{1}{p_n}(1, -a_n) \end{array} \quad (3.B)$$

Proof. We use the open cover $T^{\vec{x}} = V_1 \cup V_2$ given in the proof of 3.4, and we will show that $\widehat{\mathcal{O}}_{T^{\vec{x}}, \lambda_i}$ is the completion of $\frac{1}{p_i}(1, -a_i)$. By symmetry, we only have to consider the case $i = 1$.

For the polynomial ring $k[x_1, \dots, x_n, t]$ and the formal power series ring $k[[x_1, t]]$, consider the morphism

$$f: k[x_1, \dots, x_n, t] \rightarrow P := k[[x_1, t]]$$

of k -algebras defined by $f(t) = t$, $f(x_1) = x_1$, $f(x_2) = 1$ and $f(x_i) = (\lambda_i - x_1^{p_1})^{1/p_i}$ for $3 \leq i \leq n$, where a p_i -th root of $\lambda_i - x_1^{p_1}$ exists since k is an algebraically closed field of characteristic zero. Since f sends $x_i^{p_i} + x_1^{p_1} - \lambda_i x_2^{p_2}$ to zero for all $3 \leq i \leq n$, it induces a morphism of k -algebras $f: S[t] \rightarrow P$, and further since $f(x_2) = 1$ this induces a morphism of k -algebras

$$f: S[t]_{x_2} \rightarrow P. \quad (3.C)$$

Let $C := \frac{1}{p_1}(1, -a_1) = \langle \zeta \rangle$ be the cyclic group acting on P by $\zeta x_1 = \varepsilon x_1$ and $\zeta t = \varepsilon^{-a_1} t$ for a primitive p_1 -th root ε of unity. Certainly $f(x_i)$ with $2 \leq i \leq n$ belongs to $k[[x_1^{p_1}]] \subset P^C$. Now we claim that (3.C) induces a morphism of k -algebras

$$f: (S[t]_{x_2})_0 \rightarrow P^C. \quad (3.D)$$

If a monomial $X = x_1^{\ell_1} \dots x_n^{\ell_n} t^{\ell} \in S[t]_{x_2}$ has degree zero, then $\ell_1 \vec{x}_1 + \dots + \ell_n \vec{x}_n + \ell \vec{x} = 0$ holds. Looking at the coefficients of \vec{x}_1 , we have $\ell_1 + \ell a_1 \in p_1 \mathbb{Z}$. Thus $f(X) = x_1^{\ell_1} t^{\ell} \prod_{i=2}^n f(x_i)^{\ell_i}$ belongs to P^C , and the assertion follows.

Now let \mathfrak{m} be the maximal ideal of $(S[t]_{x_2})_0$ corresponding to λ_1 , and \mathfrak{n} be the maximal ideal of P^C . Then $\widehat{\mathcal{O}}_{T^{\vec{x}}, \lambda_1}$ is the completion of $(S[t]_{x_2})_0$ at \mathfrak{m} . Moreover $f(\mathfrak{m}) \subset \mathfrak{n}$ holds

since \mathfrak{m} is generated by monomials $x_1^{\ell_1} \cdots x_n^{\ell_n} t^\ell$ with $\ell \geq 1$ and $x_1^{p_1}/x_2^{p_2}$. Thus (3.D) induces a morphism

$$f: \widehat{\mathcal{O}}_{T^\vec{x}, \lambda_1} \rightarrow P^C. \quad (3.E)$$

To prove surjectivity, it suffices to show that f gives a surjective map $\mathfrak{m} \rightarrow \mathfrak{n}/\mathfrak{n}^2$. Since the k -vector space $\mathfrak{n}/\mathfrak{n}^2$ is spanned by monomials in P^C , it suffices to show that any monomial $x_1^{\ell_1} t^\ell$ in P^C belongs to $\text{Im } f + \mathfrak{n}^2$. Since $x_1^{\ell_1} t^\ell$ is invariant under the action of C , the coefficient of \vec{x}_1 in the normal form of $\ell_1 \vec{x}_1 + \ell \vec{x}$ is zero. Thus there exist $\ell_2 \in \mathbb{Z}$ and $\ell_3, \dots, \ell_n \in \mathbb{Z}_{\geq 0}$ such that $\ell_1 \vec{x}_1 + \dots + \ell_n \vec{x}_n + \ell \vec{x} = 0$. Now $X := x_1^{\ell_1} \cdots x_n^{\ell_n} t^\ell \in (S[t]_{x_2})_0$ satisfies

$$f(\alpha X) \equiv x_1^{\ell_1} t^\ell \pmod{\mathfrak{n}^2} \quad \text{for } \alpha := \prod_{i=3}^n \lambda_i^{-\ell_i/p_i}.$$

Hence (3.E) is surjective. Since $(S[t]_{x_2})_0$ is an algebraic normal domain by the proof of 3.4, its completion is also a normal domain by Zariski. Thus (3.E) is a surjective morphism between two-dimensional domains, and so is necessarily an isomorphism. \square

The following is now a corollary of 3.3 and 3.5.

Corollary 3.6. *If $0 \neq \vec{x} \in \mathbb{L}_+$, then $S^\vec{x}$ is a rational surface singularity.*

Proof. By 3.5 all the singularities on $T^\vec{x}$ are rational, hence there exists a resolution $f: Y \rightarrow T^\vec{x}$ such that $\mathbf{R}f_* \mathcal{O}_Y = \mathcal{O}_{T^\vec{x}}$. Since by 3.3 $T^\vec{x}$ has a tilting bundle with summand $\mathcal{O}_{T^\vec{x}}$, necessarily $H^i(\mathcal{O}_{T^\vec{x}}) = \text{Ext}_{T^\vec{x}}^i(\mathcal{O}_{T^\vec{x}}, \mathcal{O}_{T^\vec{x}}) = 0$ for all $i > 0$. By construction, we already know that $\psi_* \mathcal{O}_{T^\vec{x}} = \mathcal{O}_{S^\vec{x}}$, so this implies that $\mathbf{R}\psi_* \mathcal{O}_{T^\vec{x}} = \mathcal{O}_{S^\vec{x}}$. Composing, we see that $\mathbf{R}(\psi \circ f)_* \mathcal{O}_Y = \mathcal{O}_{S^\vec{x}}$, and so $\text{Spec } S^\vec{x}$ is rational. \square

Corollary 3.7. *If $0 \neq \vec{x} \in \mathbb{L}_+$, then the fundamental cycle associated to the morphism $\pi: Y^\vec{x} \rightarrow \text{Spec } S^\vec{x}$ is reduced.*

Proof. Since $\pi: Y^\vec{x} \rightarrow \text{Spec } S^\vec{x}$ is a resolution of a rational surface singularity, the fundamental cycle exists. Resolving the singularities in (3.B) it is clear that the dual graph of π is star shaped, with the middle curve of this star corresponding to the \mathbb{P}^1 in $T^\vec{x}$. By 3.3 the line bundle $\mathcal{L} := p^* \mathcal{O}_{\mathbb{P}^1}(1)$ on $T^\vec{x}$ satisfies $\text{Ext}_{T^\vec{x}}^1(\mathcal{L}, \mathcal{O}_{T^\vec{x}}) = 0$. It clearly has degree one on the exceptional curve. Then $\mathcal{L}_Y := \varphi^* \mathcal{L} = \mathbf{L} \varphi^* \mathcal{L}$ is a line bundle on $Y^\vec{x}$, with degree one on the middle curve and degree zero on all other curves. Furthermore

$$\begin{aligned} H^1(\mathcal{L}_Y^{-1}) &\cong \text{Ext}_{Y^\vec{x}}^1(\mathcal{L}_Y, \mathcal{O}_{Y^\vec{x}}) \\ &\cong \text{Hom}_{\text{Db}(Y^\vec{x})}(\mathbf{L} \varphi^* \mathcal{L}, \mathcal{O}_{Y^\vec{x}}[1]) \\ &\cong \text{Hom}_{\text{Db}(T^\vec{x})}(\mathcal{L}, \mathbf{R} \varphi_* \mathcal{O}_{Y^\vec{x}}[1]) \\ &\cong \text{Ext}_{T^\vec{x}}^1(\mathcal{L}, \mathcal{O}_{T^\vec{x}}) \\ &= 0. \end{aligned}$$

Since \mathcal{L}_Y has rank one, as in [W6] (see also [VdB, 3.5.4]), this implies that in the fundamental cycle of π , the middle curve has number one. Since the dual graph is star shaped, and all other self-intersection numbers are ≤ -2 , by (2.B) this then implies that the whole fundamental cycle is reduced. \square

In the sequel, we require the following description of some degenerate cases.

Lemma 3.8. *Let $0 \neq \vec{x} \in \mathbb{L}_+$ and write $\vec{x} = \sum_{i=1}^n a_i \vec{x}_i + a \vec{c}$ with $a \geq 0$ in normal form.*

- (1) *If all $a_i = 0$ (so necessarily $a > 0$), then $Y^\vec{x} = T^\vec{x} = \mathcal{O}_{\mathbb{P}^1}(-a)$ and $S^\vec{x} = k[x, y]^{\frac{1}{a}(1,1)}$.*
- (2) *If $a_i \neq 0$ and $a_j = 0$ for all $j \neq i$, then the dual graph of π in (3.A) is*

$$\begin{array}{ccccccc} \bullet & \text{---} & \bullet & \text{---} & \cdots & \text{---} & \bullet & \text{---} & \bullet \\ -1-a & & -\alpha_{i1} & & & & -\alpha_{im_i-1} & & -\alpha_{im_i} \end{array}$$

$$\text{where } \frac{p_i}{p_i - a_i} = [\alpha_{i1}, \dots, \alpha_{im_i}].$$

Proof. (1) It is well known that $S^\vec{c} = k[t_0, t_1]$, and hence $S^{a\vec{c}}$ is the a -th Veronese of $k[t_0, t_1]$, which is $k[x, y]^{\frac{1}{a}(1,1)}$. There is only a single curve above the origin in the smooth modification π , which necessarily must have negative self-intersection. Since the contraction of a $(-b)$ curve is always analytically isomorphic to $k[[x, y]]^{\frac{1}{b}(1,1)}$, this then implies

that $a = b$ and so $Y^{\vec{x}} = T^{\vec{x}} = \mathcal{O}_{\mathbb{P}^1}(-a)$.

(2) There is only one singularity in (3.B), which implies that the dual graph has the above Type A form. It is standard that α_{ij} from $\frac{p_i}{p_i - a_i} = [\alpha_{i1}, \dots, \alpha_{im_i}]$ resolves $\frac{1}{p_i}(1, -a_i)$, and thus the only thing still to be verified is the self-intersection number $-1 - a$. There are two ways of doing this: since the fundamental cycle of π is reduced by 3.7, the reconstruction algebra is easy to calculate and it can be directly verified that its quiver has the form given by intersection rules in 2.9 (which, by [W2], hold for non-minimal resolutions too). Alternatively, the number $-1 - a$ can be determined by an explicit gluing calculation on $T^{\vec{x}}$; in both cases we suppress the details. \square

3.2. Special CM Modules and the Dual Graph. Choose $0 \neq \vec{x} \in \mathbb{L}_+$. In this subsection we first give a precise criterion for when $\pi: Y^{\vec{x}} \rightarrow \text{Spec } S^{\vec{x}}$ in (3.A) is the minimal resolution, then we use the results of the previous subsections to determine the indecomposable special CM $S^{\vec{x}}$ -modules.

Lemma 3.9. *Let $0 \neq \vec{x} \in \mathbb{L}_+$. Then π is the minimal resolution if and only if $\vec{x} \notin [0, \vec{c}]$.*

Proof. Write $\vec{x} = \sum_{i=1}^n a_i \vec{x}_i + a\vec{c}$ in normal form. Since $0 \neq \vec{x} \in \mathbb{L}_+$, necessarily $a \geq 0$. Note that, by construction, the only curve in the star-shaped dual graph of π that might be a (-1) -curve is the middle one.

(\Leftarrow) Suppose that $\vec{x} \notin [0, \vec{c}]$. If all $a_i = 0$ then necessarily $a \geq 2$, and so 3.8(1) shows that π is the minimal resolution. Similarly, if $a_i \neq 0$ but $a_j = 0$ for all $j \neq i$, then the assumption $\vec{x} \notin [0, \vec{c}]$ forces $a \geq 1$, and 3.8(2) then shows that π is minimal.

Hence we can assume that $\vec{x} \notin [0, \vec{c}]$ with at least two of the a_i being non-zero. This being the case, there are at least two singular points in (3.B). By 3.7, since the fundamental cycle is reduced, the calculation (2.B) shows that middle curve then cannot be a (-1) -curve, hence the resolution is minimal.

(\Rightarrow) By contrapositive, suppose that $0 \neq \vec{x} \in [0, \vec{c}]$, say $\vec{x} = a_i \vec{x}_i$ for some i and some $0 < a_i < p_i$. Since $a = 0$, by 3.8(2) the resolution π is not minimal. \square

Hence if $x \in \mathbb{L}_+$ with $x \notin [0, \vec{c}]$, it follows that the dual graph of the minimal resolution $\pi: Y^{\vec{x}} \rightarrow \text{Spec } S^{\vec{x}}$ is (1.E), except that we have not yet determined the precise value of β . We will do this later in 4.21, since for the moment this value is not needed. As notation, for $\vec{y} \in \mathbb{L}$ write $S(\vec{y})^{\vec{x}} := \bigoplus_{i \in \mathbb{Z}} S_{\vec{y} + i\vec{x}}$.

Theorem 3.10. *Suppose that $\vec{x} \in \mathbb{L}_+$ with $\vec{x} \notin [0, \vec{c}]$, and write $\vec{x} = \sum_{i=1}^n a_i \vec{x}_i + a\vec{c}$ with $a \geq 0$ in normal form. Then*

$$\text{SCM } S^{\vec{x}} = \text{add}\{S(u\vec{x}_j)^{\vec{x}} \mid j \in [1, n], u \in I(p_j, p_j - a_j)\}.$$

Proof. The ring $S^{\vec{x}}$ has a unique singular point corresponding to the graded maximal ideal, since it is two dimensional normal and positively graded (see e.g. [P, p1]). Thus, by 2.4 we complete $S^{\vec{x}}$ at this point and pass to the formal fibre, which is still the minimal resolution. However, to aid readability, we do not add $\widehat{(\quad)}$ to the notation.

Consider the bundle $q^*\mathcal{E}$ on \mathbb{T} , and its pushdown $g_*q^*\mathcal{E}$ on $T^{\vec{x}}$. At the point λ_1 of $T^{\vec{x}}$, which is the singularity $\frac{1}{p_1}(1, -a_1)$ by 3.5, the sheaves

$$g_*q^*\mathcal{O}, g_*q^*\mathcal{O}(\vec{x}_1), \dots, g_*q^*\mathcal{O}((p_1 - 1)\vec{x}_1) \quad (3.F)$$

are all locally free away from the point λ_1 , since at any other singular point λ_i , multiplication by x_1 is invertible. Further, at the point λ_1 , (3.F) is a full list of the CM modules, indexed by the characters of $\mathbb{Z}_{p_1} = \frac{1}{p_1}(1, -a_1)$ in the obvious way. Hence by 2.14, which does not require any coprime assumption, the torsion-free pullbacks under φ of

$$\{g_*q^*\mathcal{O}(u\vec{x}_1) \mid u \in I(p_1, p_1 - a_1)\}$$

are precisely the line bundles on $Y^{\vec{x}}$ corresponding to the curves in arm 1 of the dual graph. By 3.7 they are the special bundles on $Y^{\vec{x}}$ corresponding to the curves in arm 1 of the dual graph, hence their pushdown (via π) to $S^{\vec{x}}$ are the special CM $S^{\vec{x}}$ -modules corresponding to arm 1. Since the pushdown under φ of the torsion-free pullback of φ is the identity, the pushdown to $S^{\vec{x}}$ gives the modules

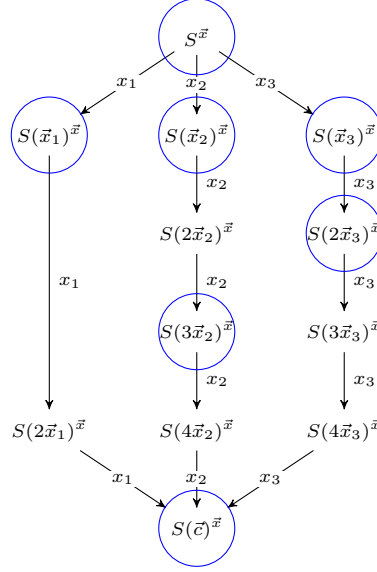
$$\{\psi_*g_*q^*\mathcal{O}(u\vec{x}_1) \mid u \in I(p_1, p_1 - a_1)\},$$

which are precisely the modules $S(u\vec{x}_1)^{\vec{x}}$. The argument for the other arms is identical. The argument for $S(\vec{c})^{\vec{x}}$ follows again by 3.7. \square

Example 3.11. Consider the example $(p_1, p_2, p_3) = (3, 5, 5)$ and $\vec{x} = 2\vec{x}_1 + 2\vec{x}_2 + 3\vec{x}_3$. The continued fractions for $\frac{p_i}{p_i - a_i}$, and the corresponding i -series are given by:

$$\begin{aligned} \frac{3}{3-2} &= [3] & 3 > 1 > 0 \\ \frac{5}{5-2} &= [2, 3] & 5 > 3 > 1 > 0 \\ \frac{5}{5-3} &= [3, 2] & 5 > 2 > 1 > 0 \end{aligned}$$

It follows from 3.10 that an additive generator of SCM R is given by the direct sum of the following circled modules:



Remark 3.12. It is in fact possible to assign each special CM $S^{\vec{x}}$ -module to its vertex in the dual graph of the minimal resolution. As in 3.11 above there are obvious irreducible morphisms between the special CM R -modules, so they must appear in the quiver of the reconstruction algebra. By the intersection theory in 2.9, we conclude that $S(\vec{c})^{\vec{x}}$ corresponds to the middle vertex, and this forces the positions of the other special CM modules relative to the dual graph.

Corollary 3.13. *If $0 \neq \vec{x} \in \mathbb{L}_+$, then the following statements hold:*

- (1) *There is an idempotent $e \in \text{End}_{\mathbb{T}}(q^*\mathcal{E})$ such that $e \text{End}_{\mathbb{T}}(q^*\mathcal{E})e \cong \text{End}_{Y^{\vec{x}}}(\mathcal{M})$.*
- (2) *There is a fully faithful embedding*

$$\text{D}^b(\text{coh } Y^{\vec{x}}) \hookrightarrow \text{D}^b(\text{coh } \mathbb{T}^{\vec{x}}).$$

Proof. (1) Consider the tilting bundle \mathcal{M} on $Y^{\vec{x}}$, generated by global sections, constructed in [VdB, 3.5.4]. Even although $\pi: Y^{\vec{x}} \rightarrow \text{Spec } S^{\vec{x}}$ need not be the minimal resolution, it is still true by [DW, 4.3] that

$$\text{End}_{Y^{\vec{x}}}(\mathcal{M}) \cong \text{End}_{S^{\vec{x}}}(\pi_*\mathcal{M})$$

On the other hand,

$$\text{End}_{\mathbb{T}}(q^*\mathcal{E}) \cong \text{End}_{S^{\vec{x}}}(\bigoplus_{\vec{y} \in [0, \vec{c}]} S(\vec{y})^{\vec{x}}).$$

As in the proof of 3.10, $\pi_*\mathcal{M}$ consists of summands of $\bigoplus_{\vec{y} \in [0, \vec{c}]} S(\vec{y})^{\vec{x}}$, hence there is an idempotent $e \in \text{End}_{\mathbb{T}}(q^*\mathcal{E})$ such that $e \text{End}_{\mathbb{T}}(q^*\mathcal{E})e \cong \text{End}_{Y^{\vec{x}}}(\mathcal{M})$.

(2) By (1), writing $A := \text{End}_{\mathbb{T}}(q^*\mathcal{E})$ then $\text{End}_{Y^{\vec{x}}}(\mathcal{M}) = eAe$, thus there is an obvious embedding of derived categories

$$\text{RHom}_{eAe}(Ae, -): \text{D}(\text{Mod } \text{End}_{Y^{\vec{x}}}(\mathcal{M})) \hookrightarrow \text{D}(\text{Mod } \text{End}_{\mathbb{T}}(q^*\mathcal{E})),$$

and also an embedding given by $-\otimes_{eAe}^{\mathbf{L}} eA$. Regardless, since $\text{gl.dim } eAe < \infty$, the above induces an embedding

$$\mathrm{D}^b(\text{mod End}_{Y^{\vec{x}}}(\mathcal{M})) \hookrightarrow \mathrm{D}^b(\text{mod End}_{\mathbb{T}}(q^*\mathcal{E})).$$

The left hand side is derived equivalent to $Y^{\vec{x}}$, and the right hand side is derived equivalent to $\mathbb{T}^{\vec{x}} := \text{Tot}(\mathcal{O}_{\mathbb{X}}(-\vec{x}))$ by 3.2, so the result follows. \square

We give a simple criterion for when the above is an equivalence later in 4.9. Note that the above result is formally very similar to the case of quotient singularities, where the reconstruction algebra embeds into the quotient stack $[k^2/G]$, but this embedding is also very rarely an equivalence.

4. CATEGORICAL EQUIVALENCES

In this section we investigate the conditions on \vec{x} under which

$$\text{coh } \mathbb{X} \simeq \text{qgr}^{\mathbb{Z}} S^{\vec{x}} \simeq \text{qgr}^{\mathbb{Z}} \Gamma_{\vec{x}}$$

holds. This allows us, in 4.9, to give a precise criterion for when the embedding in 3.13(2) is an equivalence, and further it allows us in 4.21 to determine the middle self-intersection number in (1.E). Throughout many results in this section, a coprime condition $(p_i, a_i) = 1$ naturally appears, and in §4.3 we show that we can always change parameters so that this coprime condition holds.

4.1. General Results on Categorical Equivalences. To simplify the notation, in this subsection we first produce categorical equivalences in a very general setting, before specialising in the next subsection to the case of the weighted projective line.

We start with a basic observation. Let G be an abelian group and A a noetherian G -graded k -algebra. For an idempotent $e \in A_0$, $B := eAe$ is a noetherian G -graded k -algebra. The functor

$$E := e(-) : \text{mod}^G A \rightarrow \text{mod}^G B$$

has a left adjoint functor E_{λ} and a right adjoint functor E_{ρ} given by

$$E_{\lambda} := Ae \otimes_B - : \text{mod}^G B \rightarrow \text{mod}^G A$$

$$E_{\rho} := \text{Hom}_B(eA, -) : \text{mod}^G B \rightarrow \text{mod}^G A.$$

Moreover $EE_{\lambda} = \text{id}_{\text{mod}^G B} = EE_{\rho}$ holds, and for the natural morphism $m : Ae \otimes_B eA \rightarrow A$, the counit $\eta : E_{\lambda}E \rightarrow \text{id}_{\text{mod}^G A}$ is given by $m \otimes_A -$ and the unit $\varepsilon : \text{id}_{\text{mod}^G A} \rightarrow E_{\rho}E$ is given by $\text{Hom}_A(m, -)$.

The following basic observation is a prototype of our results in this subsection.

Proposition 4.1. *If $\dim_k A/(e) < \infty$, then E induces an equivalence $\text{qgr}^G A \simeq \text{qgr}^G B$.*

Proof. Clearly E_{λ} and E induce an adjoint pair $E_{\lambda} : \text{qgr}^G A \rightarrow \text{qgr}^G B$ and $E : \text{qgr}^G B \rightarrow \text{qgr}^G A$. For any $X \in \text{mod}^G A$, both the kernel and cokernel of $m \otimes_A X : E_{\lambda}EX \rightarrow X$ are finite dimensional since they are finitely generated $(A/(e))$ -modules. Therefore E_{λ} and E give the desired equivalences. \square

In the rest of this subsection, let G be an abelian group and H a subgroup of G of finite index. Assume that A is a noetherian G -graded k -algebra, and let $B := A^H = \bigoplus_{g \in H} A_g$ be the H -Veronese subring of A .

Lemma 4.2. *B is a noetherian k -algebra and A is a finitely generated B -module.*

Proof. There is a finite direct sum decomposition $A = \bigoplus_{g \in G/H} A(g)^H$ as B -modules. For any submodule M of $A(g)^H$, it is easy to check that the ideal AM of A satisfies $AM \cap A(g)^H = M$. Therefore $A(g)^H$ is a noetherian B -module, since A is a noetherian ring. The assertion follows. \square

We say that $X \in \text{mod}^G A$ has *depth at least two* if $\text{Ext}_A^i(Y, X) = 0$ for any $i = 0, 1$ and $Y \in \text{mod}^G A$ with $\dim_k Y < \infty$. We denote by $\text{mod}_2^G A$ the full subcategory of $\text{mod}^G A$ consisting of modules with depth at least two. We define $\text{mod}_2^H B$ similarly.

Theorem 4.3. *Let G be an abelian group, H a subgroup of G of finite index, A a noetherian G -graded k -algebra, and $B := A^H$. Then the following conditions are equivalent.*

- (1) *The natural functor $(-)^H: \mathbf{qgr}^G A \rightarrow \mathbf{qgr}^H B$ is an equivalence.*
- (2) *For any $i \in G$, the ideal $I^i := A(i)^H \cdot A(-i)^H$ of B satisfies $\dim_k(B/I^i) < \infty$.*

If A belongs to $\mathbf{mod}_2^G A$, then the following condition is also equivalent.

- (3) *The natural functor $(-)^H: \mathbf{mod}_2^G A \rightarrow \mathbf{mod}_2^H B$ is an equivalence.*

Proof. Consider the matrix algebra

$$C = (A(i-j)^H)_{i,j \in G/H}$$

whose rows and columns are indexed by G/H , and the product is given by the matrix multiplication together with the product in A :

$$(s_{i,j}) \cdot (t_{i,j}) := \left(\sum_{k \in G/H} s_{i,k} \cdot t_{k,j} \right).$$

Now we fix a complete set I of representatives of G/H in G . Then C has an H -grading given by

$$C_h := (A_{i-j+h})_{i,j \in I}.$$

By [IL, Theorem 3.1] there is an equivalence

$$F: \mathbf{mod}^G A \simeq \mathbf{mod}^H C \tag{4.A}$$

sending $M = \bigoplus_{i \in G} M_i$ to $F(M) = \bigoplus_{h \in H} F(M)_h$, where $F(M)_h$ is defined by

$$F(M)_h := (M_{i+h})_{i \in I}$$

and the C -module structure is given by

$$(s_{i,j})_{i,j \in I} \cdot (m_i)_{i \in I} := \left(\sum_{j \in I} s_{i,j} \cdot m_j \right)_{i \in I}.$$

On the other hand, let $e \in C$ be the idempotent corresponding to $0 \in G/H$. Since $eCe = B$ holds, there is an exact functor

$$E := e(-): \mathbf{mod}^H C \rightarrow \mathbf{mod}^H B \tag{4.B}$$

such that the following diagram commutes:

$$\begin{array}{ccc} \mathbf{mod}^G A & \xrightarrow{(-)^H} & \mathbf{mod}^H B \\ & \searrow \scriptstyle F & \nearrow \scriptstyle E \\ & \mathbf{mod}^H C & \end{array}$$

The functor (4.B) has a left adjoint functor $E_\lambda := Ce \otimes_B -: \mathbf{mod}^H B \rightarrow \mathbf{mod}^H C$ and a right adjoint functor $E_\rho := \mathrm{Hom}_B(eC, -): \mathbf{mod}^H B \rightarrow \mathbf{mod}^H C$.

(1) \Leftrightarrow (2) The functors F and E induce an equivalence $F: \mathbf{qgr}^G A \simeq \mathbf{qgr}^H C$ and a functor

$$E: \mathbf{qgr}^H C \rightarrow \mathbf{qgr}^H B \tag{4.C}$$

respectively, which make the following diagram commutative:

$$\begin{array}{ccc} \mathbf{qgr}^G A & \xrightarrow{(-)^H} & \mathbf{qgr}^H B \\ & \searrow \scriptstyle F & \nearrow \scriptstyle E \\ & \mathbf{qgr}^H C & \end{array}$$

Thus the functor $(-)^H: \mathbf{qgr}^G A \rightarrow \mathbf{qgr}^H B$ is an equivalence if and only if the functor (4.C) is an equivalence. The functor $E_\lambda: \mathbf{mod}^H B \rightarrow \mathbf{mod}^H C$ induces a left adjoint functor $E_\lambda: \mathbf{qgr}^H B \rightarrow \mathbf{qgr}^H C$ of (4.C). Clearly $EE_\lambda = \mathrm{id}_{\mathbf{qgr}^H B}$ holds, and the counit $\eta: E_\lambda E \rightarrow \mathrm{id}_{\mathbf{qgr}^H C}$ is given by $m \otimes_C -$, where m is the natural morphism

$$m: Ce \otimes_B eC \rightarrow C. \tag{4.D}$$

Thus the condition (1) holds if and only if η is an isomorphism of functors if and only if m is an isomorphism in $\mathbf{qgr}^H C$. On the other hand, the cokernel of m is $C/(e)$, where (e) is the two-sided ideal of C generated by e , and the kernel of m is a finitely generated

$C/(e)$ -module. Therefore (1) holds if and only if the factor algebra $C/(e)$ of C is finite dimensional if and only if (2) holds, by the following observation.

Lemma 4.4. $\dim_k C/(e) < \infty$ if and only if the condition (2) holds.

Proof. Since

$$C/(e) = (A(i-j)^H / (A(i)^H \cdot A(-j)^H))_{i,j \in I}$$

holds, $C/(e)$ is finite dimensional if and only if $A(i-j)^H / (A(i)^H \cdot A(-j)^H)$ is finite dimensional for any $i, j \in I$. This implies the condition (2) by considering the case $i = j$.

Conversely assume that (2) holds. Since there is a surjective map

$$A(i-j)^H \otimes_B \frac{B}{A(j)^H \cdot A(-j)^H} = \frac{A(i-j)^H}{A(i-j)^H \cdot A(j)^H \cdot A(-j)^H} \rightarrow \frac{A(i-j)^H}{A(i)^H \cdot A(-j)^H}$$

whose domain is finite dimensional, the target is also finite dimensional. Thus the assertion holds. \square

(2) \Leftrightarrow (3) Assume that $A \in \text{mod}_2^G A$. Clearly the equivalence (4.A) induces equivalences

$$F: \text{mod}_0^G A \simeq \text{mod}_0^H C \quad \text{and} \quad F: \text{mod}_2^G A \simeq \text{mod}_2^H C.$$

The remainder of the proof requires the following general lemma.

Lemma 4.5. *With the setup as above,*

(1) *The functor (4.B) induces a functor*

$$E: \text{mod}_2^H C \rightarrow \text{mod}_2^H B. \quad (4.E)$$

(2) *The functor $E_\rho: \text{mod}^H B \rightarrow \text{mod}^H C$ induces a functor $E_\rho: \text{mod}_2^H B \rightarrow \text{mod}_2^H C$.*

(3) *$X \in \text{mod}^H C$ belongs to $\text{mod}_0^H C$ if and only if $\text{Ext}_C^i(X, \text{mod}_2^H C) = 0$ for $i = 0, 1$.*

Proof. (1) Let $X \in \text{mod}_2^H C$, $Y \in \text{mod}_0^H B$ and $\mathbf{E}_\lambda Y := Ce \otimes_B^{\mathbf{L}} Y$. Since $H^i(\mathbf{E}_\lambda Y)$ is zero for any $i > 0$ and belongs to $\text{mod}_0^H C$ for any $i \leq 0$, we have $\text{Hom}_{\text{D}^b(\text{mod } C)}(\mathbf{E}_\lambda Y, X[i]) = 0$ for $i = 0, 1$. Using $\mathbf{R}\text{Hom}_B(Y, EX) = \mathbf{R}\text{Hom}_C(\mathbf{E}_\lambda Y, X)$, we have $\text{Ext}_B^i(Y, EX) = 0$ for $i = 0, 1$.

(2) Let $X \in \text{mod}_2^H B$, $Y \in \text{mod}_0^H C$ and $\mathbf{E}_\rho X := \mathbf{R}\text{Hom}_B(eC, X)$. Since $EY \in \text{mod}_0^H B$ and $\mathbf{R}\text{Hom}_C(Y, \mathbf{E}_\rho X) = \mathbf{R}\text{Hom}_B(EY, X)$ hold, we have $\text{Hom}_{\text{D}^b(\text{mod } C)}(Y, \mathbf{E}_\rho X[i]) = 0$ for $i = 0, 1$. There is a triangle $E_\rho X \rightarrow \mathbf{E}_\rho X \rightarrow Z \rightarrow E_\rho X[1]$ satisfying $H^i(Z) = 0$ for all $i \leq 1$. Applying $\text{Hom}_{\text{D}^b(\text{mod } C)}(Y, -)$ gives $\text{Ext}_C^i(Y, E_\rho X) = 0$ for $i = 0, 1$.

(3) Our assumption $A \in \text{mod}_2^G A$ implies $C = \bigoplus_{i \in I} FA(i) \in \text{mod}_2^H C$. Let $0 \rightarrow T \rightarrow X \rightarrow F \rightarrow 0$ and $0 \rightarrow \Omega F \rightarrow P \rightarrow F \rightarrow 0$ be exact sequences in $\text{mod}^H C$ such that T is the largest submodule of X which belongs to $\text{mod}_0^H C$ and P is an H -graded projective C -module. Then ΩF belongs to $\text{mod}_2^H C$ since $C \in \text{mod}_2^H C$. Applying $\text{Hom}_C(-, \Omega F)$ to the first sequence gives an exact sequence

$$0 = \text{Hom}_C(T, \Omega F) \rightarrow \text{Ext}_C^1(F, \Omega F) \rightarrow \text{Ext}_C^1(X, \Omega F) = 0.$$

Thus $\text{Ext}_C^1(F, \Omega F) = 0$ holds, and F is projective in $\text{mod}^H C$. Hence $X = T \oplus F$, and so $\text{Hom}_C(X, C) = 0$ implies that $F = 0$. Therefore $X = T$ belongs to $\text{mod}_0^H C$. \square

It follows from 4.5 that there is a commutative diagram

$$\begin{array}{ccc} \text{mod}_2^G A & \xrightarrow{(-)^H} & \text{mod}_2^H B \\ & \searrow \scriptstyle F & \nearrow \scriptstyle E \\ & \text{mod}_2^H C & \end{array}$$

Thus the functor $(-)^H: \text{mod}_2^G A \rightarrow \text{mod}_2^H B$ is an equivalence if and only if the functor (4.E) is an equivalence. By 4.5(2), there is a right adjoint functor $E_\rho: \text{mod}_2^H B \rightarrow \text{mod}_2^H C$ of (4.E). Clearly $EE_\rho = \text{id}_{\text{mod}_2^H B}$ holds, and the unit $\varepsilon: \text{id}_{\text{mod}_2^H C} \rightarrow E_\rho E$ is given by $\text{Hom}_C(m, -)$, where m is the morphism (4.D). Thus the condition (3) holds if and only if $\varepsilon = \text{Hom}_C(m, -)$ is an isomorphism of functors.

Now fix $X \in \text{mod}_2^H C$ and apply $\text{Hom}_C(-, X)$ to exact sequences $0 \rightarrow (e) \rightarrow C \rightarrow C/(e) \rightarrow 0$ and $0 \rightarrow \text{Ker } m \rightarrow Ce \otimes_B eC \rightarrow (e) \rightarrow 0$. This gives exact sequences

$$\begin{aligned} 0 \rightarrow \text{Hom}_C(C/(e), X) \rightarrow X \rightarrow \text{Hom}_C((e), X) \rightarrow \text{Ext}_C^1(C/(e), X) \rightarrow 0 \\ 0 \rightarrow \text{Hom}_C((e), X) \rightarrow \text{Hom}_C(Ce \otimes_B eC, X) \rightarrow \text{Hom}_C(\text{Ker } m, X). \end{aligned}$$

Therefore, if $C/(e)$ is finite dimensional, then so is $\text{Ker } m$ and hence ε is an isomorphism. Conversely, if ε is an isomorphism, then $\text{Ext}_C^i(C/(e), X) = 0$ for $i = 0, 1$ for any $X \in \text{mod}_2^H C$ and hence $C/(e)$ is finite dimensional by 4.5(3). Consequently (3) is equivalent to (2), again by 4.4. \square

Later we need the following observation.

Lemma 4.6. *In the setting of 4.3, assume that the condition (2) is satisfied. Then for any $X \in \text{mod}^G A$ and $Y \in \text{mod}_2^G A$, there is an isomorphism*

$$\text{Hom}_B(X^H, Y^H) \cong \text{Hom}_A(X, Y)^H$$

of H -graded k -modules.

Proof. Clearly $\text{Hom}_B(X^H, Y^H) = \text{Hom}_B(EFX, EFX) = \text{Hom}_C(E_\lambda EFX, FY)$. This is isomorphic to $\text{Hom}_C(FX, FY) = \text{Hom}_A(X, Y)^H$ since the kernel and the cokernel of $\eta_X: E_\lambda EX \rightarrow X$ are finite dimensional by our assumptions. \square

4.2. Categorical Equivalences for Weighted Projective Lines. In this subsection, we apply the general results of the previous subsection to describe the precise conditions on $\vec{x} \in \mathbb{L}$ for which $\text{qgr}^{\mathbb{Z}} S^{\vec{x}} \simeq \text{coh } \mathbb{X}$ holds. As before, write $S(\vec{y})^{\vec{x}} := \bigoplus_{i \in \mathbb{Z}} S_{\vec{y}+i\vec{x}}$. This subsection does not require the condition that \vec{x} belongs to \mathbb{L}_+ . Instead we assume that \vec{x} is not torsion. Then $\mathbb{Z}\vec{x}$ is a subgroup of \mathbb{L} of finite index, and the following observation holds by 4.2.

Lemma 4.7. *If $\vec{x} \in \mathbb{L}$ is not torsion, then S is a finitely generated $S^{\vec{x}}$ -module.*

The following is the main result in this subsection, where a special case $\vec{x} = \vec{\omega}$ was given in [GL2]. Another approach can be found in [H1].

Theorem 4.8. *Suppose that $\vec{x} = \sum_{i=1}^n a_i \vec{x}_i + a\vec{c} \in \mathbb{L}$ is not torsion, and denote $R := S^{\vec{x}}$. Then the following conditions are equivalent.*

- (1) *The natural functor $(-)^{\vec{x}}: \text{CM}^{\mathbb{L}} S \rightarrow \text{CM}^{\mathbb{Z}} R$ is an equivalence.*
- (2) *The natural functor $(-)^{\vec{x}}: \text{qgr}^{\mathbb{L}} S \rightarrow \text{qgr}^{\mathbb{Z}} R$ is an equivalence.*
- (3) *For any $\vec{z} \in \mathbb{L}$, the ideal $I^{\vec{z}} := S(\vec{z})^{\vec{x}} \cdot S(-\vec{z})^{\vec{x}}$ of R satisfies $\dim_k(R/I^{\vec{z}}) < \infty$.*
- (4) *$(p_i, a_i) = 1$ for all $1 \leq i \leq n$.*

Proof. (1) \Leftrightarrow (2) \Leftrightarrow (3) These are shown in 4.3 since $\text{CM}^{\mathbb{L}} S = \text{mod}_2^{\mathbb{L}} S$ and $\text{CM}^{\mathbb{Z}} R = \text{mod}_2^{\mathbb{Z}} R$. (3) \Rightarrow (4). By contrapositive, assume that a_1 and p_1 are not coprime. Then the normal form of any element in $\vec{x}_1 + \mathbb{Z}\vec{x}$ (respectively, $-\vec{x}_1 + \mathbb{Z}\vec{x}$) contains a positive multiple of \vec{x}_1 . Thus we have

$$I^{\vec{x}_1} \subset Sx_1 \cdot Sx_1 = Sx_1^2.$$

Therefore the condition (3) implies that the algebra $R/(R \cap Sx_1^2)$ is finite dimensional. Since S/Sx_1^2 is a finitely generated $R/(R \cap Sx_1^2)$ -module by 4.7, it is also finite dimensional. This is a contradiction since S has Krull dimension two.

(4) \Rightarrow (3). Assume that $(p_i, a_i) = 1$ for all i . If $R/I^{\vec{y}}$ and $R/I^{\vec{z}}$ are finite dimensional, then so is $R/I^{\vec{y}+\vec{z}}$ since $I^{\vec{y}} \cdot I^{\vec{z}} \subset I^{\vec{y}+\vec{z}}$ holds. Thus we only have to show that $R/I^{\vec{x}_i}$ is finite dimensional for each i with $1 \leq i \leq n$. We will show that $I^{\vec{x}_i}$ contains a certain power A of x_i and a certain monomial B of x_j 's with $j \neq i$. Then it is easy to check that $S/(SA + SB)$ is finite dimensional, and hence $R/(RA + RB) = (S/(SA + SB))^{\vec{x}}$ and $R/I^{\vec{x}_i}$ are also finite dimensional.

For the least common multiple p of p_1, \dots, p_n , we have $p\vec{x} = q\vec{c}$ for some $q > 0$. Then

$$I^{\vec{x}_i} = S(\vec{x}_i)^{\vec{x}} \cdot S(-\vec{x}_i)^{\vec{x}} \supset S_{\vec{x}_i} \cdot S_{-\vec{x}_i+p\vec{x}} \ni x_i \cdot x_i^{p_i q - 1} = x_i^{p_i q}.$$

Thus $I^{\vec{x}_i}$ contains a power of x_i . On the other hand, since a_i and p_i are coprime, there exist integers ℓ and m such that $a_i \ell + 1 = p_i m$. Then the normal form of $\vec{x}_i + \ell\vec{x}$ does not contain a positive multiple of \vec{x}_i , and hence $S(\vec{x}_i)^{\vec{x}} \supset S_{\vec{x}_i+\ell\vec{x}}$ contains a monomial of x_j 's

with $j \neq i$. Applying a similar argument to $S(-\vec{x}_i)^{\vec{x}}$, we have that $I^{\vec{x}_i} = S(\vec{x}_i)^{\vec{x}} \cdot S(-\vec{x}_i)^{\vec{x}}$ contains a monomial of x_j 's with $j \neq i$. Thus the assertion follows. \square

The following is a geometric corollary of the results in this subsection.

Corollary 4.9. *Suppose that $0 \neq \vec{x} \in \mathbb{L}_+$ and write $\vec{x} = \sum_{i=1}^n a_i \vec{x}_i + a\vec{c}$ in normal form. If $n \geq 1$ and $(p_i, a_i) = 1$ for all $1 \leq i \leq n$, then the fully faithful embedding*

$$D^b(\text{coh } Y^{\vec{x}}) \hookrightarrow D^b(\text{coh } \mathbb{T}^{\vec{x}})$$

in 3.13 is an equivalence if and only if every $a_i = 1$, that is $\vec{x} = \sum_{i=1}^n x_i + a\vec{c}$.

Proof. We use the notation from the proof of 3.13. Note that from the assumption $(p_i, a_i) = 1$ for every $1 \leq i \leq n$, necessarily each a_i is non-zero. Next, the indecomposable summands of $\pi_* \mathcal{M}$ are pairwise non-isomorphic by combining [VdB, 3.5.3] and [DW, 4.3], and the summands of $\bigoplus_{\vec{y} \in [0, \vec{c}]} S(\vec{y})^{\vec{x}}$ are pairwise non-isomorphic by 4.8(1).

The embedding in 3.13 is induced from idempotents using the observation that $\pi_* \mathcal{M}$ is a summand of $\bigoplus_{\vec{y} \in [0, \vec{c}]} S(\vec{y})^{\vec{x}}$. It follows that the embedding is an equivalence if and only for all $t = 1, \dots, n$, the i -series on arm t has maximum length. By 2.13 this holds if and only if every $a_i = 1$. \square

4.3. Changing Parameters. Our next main result, 4.12, shows that up to a change of parameters, we can always assume the condition $(p_i, a_i) = 1$ for all $1 \leq i \leq n$ that appears in both 4.8(4) and 4.9. This requires the following lemma.

Lemma 4.10. *Let $\vec{x} = \sum_{i=1}^n a_i \vec{x}_i + a\vec{c}$ be in normal form.*

- (1) *We have $S_{\vec{x}} = (\prod_{i=1}^n x_i^{a_i}) S_{a\vec{c}}$ and $S_{\vec{x}+m\vec{c}} = S_{\vec{x}} \cdot S_{m\vec{c}}$ for all $m \geq 0$.*
- (2) *$S_{a\vec{c}}$ is an $(a+1)$ -dimensional vector space, and for any $j \neq k$ a basis of $S_{a\vec{c}}$ is given by $t_0^\ell t_1^{a-\ell}$ with $0 \leq \ell \leq a$.*

Proof. Both assertions are elementary, see [GL1]. \square

We now fix notation. Let $S := S_{\mathbf{p}, \boldsymbol{\lambda}}$, and fix a subset I of $\{1, \dots, n\}$. For each $i \in I$, choose a divisor d_i of p_i . Let $\mathbf{p}_i := p_i/d_i$, $\mathbf{p}' := (\mathbf{p}_i \mid i \in I)$, $\boldsymbol{\lambda}' := (\lambda_i \mid i \in I)$,

$$S' := S_{\mathbf{p}', \boldsymbol{\lambda}'} = \frac{k[t_0, t_1, x_i \mid i \in I]}{(x_i^{\mathbf{p}_i} - \ell_i(t_0, t_1) \mid i \in I)}$$

and $\mathbb{L}' := \mathbb{L}(\mathbf{p}_i \mid i \in I) = \langle \vec{x}_i, \vec{c} \mid i \in I \rangle / (\mathbf{p}_i \vec{x}_i - \vec{c} \mid i \in I)$.

Proposition 4.11. *With notation as above,*

- (1) *There is a monomorphism $\iota: \mathbb{L}' \rightarrow \mathbb{L}$ of groups sending \vec{x}_i to $d_i \vec{x}_i$ for each $i \in I$ and \vec{c} to \vec{c} .*
- (2) *There is a monomorphism $S' \rightarrow S$ of k -algebras sending x_i to $x_i^{d_i}$ for each $i \in I$ and t_j to t_j for $j = 0, 1$, which induces an isomorphism $S' \simeq \bigoplus_{\vec{x} \in \mathbb{L}'} S_{\iota(\vec{x})}$.*
- (3) *Let $\vec{x} \in \mathbb{L}$ be an element with normal form $\vec{x} = \sum_{i \in I} a_i \vec{x}_i + a\vec{c}$ such that a_i is a multiple of d_i . For $\mathbf{a}_i := a_i/d_i$ and $\vec{x} := \sum_{i \in I} \mathbf{a}_i \vec{x}_i + a\vec{c} \in \mathbb{L}'$, we have $(S')^{\vec{x}} = S^{\vec{x}}$.*

Proof. (1) Clearly ι is well-defined. Assume that $\vec{x} \in \mathbb{L}'$ with normal form $\vec{x} = \sum_{i \in I} a_i \vec{x}_i + a\vec{c}$ belongs to the kernel of ι . Then $0 = \iota(\vec{x}) = \sum_{i \in I} a_i d_i \vec{x}_i + a\vec{c}$, where the right hand side is a normal form in \mathbb{L} , and so $a_i = 0 = a$ for all i . Hence $\vec{x} = 0$.

(2) Take any element $\vec{x} \in \mathbb{L}'$ with a normal form $\vec{x} = \sum_{i \in I} \mathbf{a}_i \vec{x}_i + a\vec{c}$. Then by 4.10(1)(2), $S'_{\vec{x}}$ has a k -basis

$$t_0^j t_1^{a-j} \prod_{i \in I} x_i^{a_i} \quad 0 \leq j \leq a.$$

Since $\iota(\vec{x})$ has a normal form $\sum_{i \in I} a_i d_i \vec{x}_i + a\vec{c}$, it follows from 4.10(1)(2) that $S_{\iota(\vec{x})}$ has a k -basis $t_0^j t_1^{a-j} \prod_{i \in I} x_i^{a_i d_i}$ for $0 \leq j \leq a$. The assertion follows.

(3) Immediate from (2). \square

Proposition 4.12. *Suppose that $\vec{x} \in \mathbb{L}$ is not torsion, and write $\vec{x} = \sum_{i=1}^n a_i \vec{x}_i + a\vec{c} \in \mathbb{L}$ in normal form. Let $I := \{1 \leq i \leq n \mid a_i \neq 0\}$, and we consider the parameters $(\mathbf{p}', \boldsymbol{\lambda}')$ defined by $\mathbf{p}' := (\mathbf{p}_i \mid i \in I)$ for $\mathbf{p}_i := p_i/(a_i, p_i)$ and $\boldsymbol{\lambda}' := (\lambda_i \mid i \in I)$. As above, set $\vec{x} := \sum_{i \in I} \mathbf{a}_i \vec{x}_i + a\vec{c} \in \mathbb{L}'$, then the following statements hold.*

- (1) *There is an isomorphism $S_{\mathbf{p},\lambda}^{\vec{x}} \cong S_{\mathbf{p}',\lambda'}^{\vec{x}}$ as \mathbb{Z} -graded k -algebras.*
- (2) *There are equivalences $\mathrm{CM}^{\mathbb{Z}} S_{\mathbf{p},\lambda}^{\vec{x}} \simeq \mathrm{CM}^{\mathbb{Z}} S_{\mathbf{p}',\lambda'}^{\vec{x}} \simeq \mathrm{CM}^{\mathbb{L}} S_{\mathbf{p}',\lambda'}^{\vec{x}}$.*
- (3) *There are equivalences $\mathrm{qgr}^{\mathbb{Z}} S_{\mathbf{p},\lambda}^{\vec{x}} \simeq \mathrm{qgr}^{\mathbb{Z}} S_{\mathbf{p}',\lambda'}^{\vec{x}} \simeq \mathrm{coh} \mathbb{X}_{\mathbf{p}',\lambda'}$.*

Proof. Part (1) follows directly from 4.11(3). Certainly this induces the left equivalences in (2) and (3). Applying 4.8 to $S_{\mathbf{p}',\lambda'}^{\vec{x}}$ gives the right equivalences in (2) and (3). \square

Thus we can always replace $\mathbb{X}_{\mathbf{p},\lambda}$ by some equivalent $\mathbb{X}_{\mathbf{p}',\lambda'}$ for which the coprime assumptions in both 4.8(4) and 4.9 hold. Note also that the above implies that qgr of Veronese subrings (with $0 \neq \vec{x} \in \mathbb{L}_+$) of weighted projective lines always give weighted projective lines, but maybe with different parameters.

4.4. Algebraic Approach to Special CM Modules. In this subsection we give an algebraic treatment of the special CM $S^{\vec{x}}$ -modules, and show how to determine the rank one special CM modules without assuming any of the geometry. Hence this subsection is independent of §3, and the techniques developed will be used later to obtain geometric corollaries. Note however that the geometry is required to deduce that there are no higher rank indecomposable special CM modules; this algebraic approach seems only to be able to deal with the rank one specials.

Consider $\mathbb{X}_{\mathbf{p},\lambda}$ and let $\vec{x} \in \mathbb{L}$ be an element with normal form $\vec{x} = \sum_{i=1}^n a_i \vec{x}_i + a\vec{c}$ with $a \geq 0$. By 4.12 we can assume, by changing parameters if necessary, that $(a_i, p_i) = 1$ for all $1 \leq i \leq n$. As before, let $R := S^{\vec{x}}$, then by 4.8 there is an equivalence

$$(-)^{\vec{x}}: \mathrm{CM}^{\mathbb{L}} S \rightarrow \mathrm{CM}^{\mathbb{Z}} R.$$

Below we will often use the identification

$$S_{\vec{y}-\vec{x}} \cong \mathrm{Hom}_S^{\mathbb{L}}(S(\vec{x}), S(\vec{y})) \quad (4.F)$$

for any $\vec{x}, \vec{y} \in \mathbb{L}$. Recall that the AR translation functor of R is given by

$$\tau_R := \mathrm{Hom}_R(-, \omega_R) \circ \mathrm{Hom}_R(-, R): \mathrm{CM}^{\mathbb{Z}} R \rightarrow \mathrm{CM}^{\mathbb{Z}} R,$$

where ω_R is the \mathbb{Z} -graded canonical module of R .

Proposition 4.13. *With the setup as above, the following statements hold.*

- (1) *There is an isomorphism $\omega_R \cong S(\vec{\omega})^{\vec{x}}$.*
- (2) *There is a commutative diagram*

$$\begin{array}{ccc} \mathrm{CM}^{\mathbb{L}} S & \xrightarrow{(-)^{\vec{x}}} & \mathrm{CM}^{\mathbb{Z}} R \\ \tau_S = (\vec{\omega}) \downarrow & & \downarrow \tau_R \\ \mathrm{CM}^{\mathbb{L}} S & \xrightarrow{(-)^{\vec{x}}} & \mathrm{CM}^{\mathbb{Z}} R. \end{array} \quad (4.G)$$

Proof. (1) Using 4.6, we have $\mathrm{Ext}_R^i(k, S(\vec{\omega})^{\vec{x}}) = \mathrm{Ext}_S^i(k, S(\vec{\omega}))^{\vec{x}}$, which is k for $i = 2$ and zero for $i \neq 2$. Thus $S(\vec{\omega})^{\vec{x}}$ is the \mathbb{Z} -graded canonical module of R .

(2) Let $X \in \mathrm{CM}^{\mathbb{L}} S$. Using (1) and 4.6,

$$\tau_R(X^{\vec{x}}) = \mathrm{Hom}_R(\mathrm{Hom}_R(X^{\vec{x}}, S^{\vec{x}}), S(\vec{\omega})^{\vec{x}}) = \mathrm{Hom}_S((X, S), S(\vec{\omega}))^{\vec{x}} = X(\vec{\omega})^{\vec{x}}. \quad \square$$

The following gives an algebraic criterion for certain CM R -modules to be special.

Lemma 4.14. *For $\vec{y} \in \mathbb{L}$, the CM R -module $S(\vec{y})^{\vec{x}}$ is special if and only if*

$$S_{\vec{y}+\vec{\omega}+\ell\vec{x}} = \sum_{m \in \mathbb{Z}} S_{\vec{\omega}+m\vec{x}} \cdot S_{\vec{y}+(\ell-m)\vec{x}} \quad (4.H)$$

holds for all $\ell \in \mathbb{Z}$.

Proof. As above write $\tau_R: \mathrm{CM}^{\mathbb{Z}} R \simeq \mathrm{CM}^{\mathbb{Z}} R$ for the AR -translation. This yields AR duality

$$D \mathrm{Ext}_{\mathrm{mod}^{\mathbb{Z}} R}^1(X, Y) \simeq \mathrm{Hom}_{\overline{\mathrm{CM}}^{\mathbb{Z}} R}(Y, \tau_R X) \quad (4.I)$$

for any $X, Y \in \mathrm{CM}^{\mathbb{Z}} R$, where $\overline{\mathrm{CM}}^{\mathbb{Z}} R$ is the quotient category of $\mathrm{CM}^{\mathbb{Z}} R$ by the ideal generated by $\{\omega_R(\ell) \mid \ell \in \mathbb{Z}\}$. By 4.13(1), $S(\vec{\omega} + \ell\vec{x})^{\vec{x}} = \omega_R(\ell)$ holds, and hence there is an induced equivalence

$$(-)^{\vec{x}}: (\mathrm{CM}^{\mathbb{L}} S)/I \simeq \overline{\mathrm{CM}}^{\mathbb{Z}} R \quad (4.J)$$

for the ideal I of the category $\mathbf{CM}^{\mathbb{L}}S$ generated by $\text{add}\{S(\vec{\omega} + \ell\vec{x}) \mid \ell \in \mathbb{Z}\}$. It follows that

$$\begin{aligned} D\text{Ext}_R^1(S(\vec{y})^{\vec{x}}, R) &= \bigoplus_{\ell \in \mathbb{Z}} D\text{Ext}_{\text{mod}^{\mathbb{Z}}R}^1(S(\vec{y})^{\vec{x}}, R(\ell)) \\ &\stackrel{(4.1)}{\cong} \bigoplus_{\ell \in \mathbb{Z}} \text{Hom}_{\overline{\mathbf{CM}}^{\mathbb{Z}}R}(R, (\tau_R(S(\vec{y})^{\vec{x}}))(\ell)) \\ &\stackrel{(4.G)(4.J)}{\cong} \bigoplus_{\ell \in \mathbb{Z}} \frac{\text{Hom}_{\mathbf{CM}^{\mathbb{L}}S}(S, S(\vec{y} + \vec{\omega} + \ell\vec{x}))}{I(S, S(\vec{y} + \vec{\omega} + \ell\vec{x}))}. \end{aligned}$$

Thus $S(\vec{y})^{\vec{x}}$ is special if and only if $\text{Hom}_{\mathbf{CM}^{\mathbb{L}}S}(S, S(\vec{y} + \vec{\omega} + \ell\vec{x})) = I(S, S(\vec{y} + \vec{\omega} + \ell\vec{x}))$ holds for all $\ell \in \mathbb{Z}$. Since $\text{Hom}_{\mathbf{CM}^{\mathbb{L}}S}(S, S(\vec{y} + \vec{\omega} + \ell\vec{x})) = S_{\vec{y} + \vec{\omega} + \ell\vec{x}}$ and $I(S, S(\vec{y} + \vec{\omega} + \ell\vec{x})) = \sum_{m \in \mathbb{Z}} S_{\vec{\omega} + m\vec{x}} \cdot S_{\vec{y} + (\ell - m)\vec{x}}$ hold by (4.F), the assertion follows. \square

We also require the following, which is much more elementary.

Lemma 4.15 ([GL1]). *Suppose that $x \in \mathbb{L}$.*

- (1) *If $\vec{y} \in \mathbb{L}$ with $\vec{x} - \vec{y} \in \mathbb{L}_+$, write $\vec{y} = \sum_{i=1}^n b_i \vec{x}_i + b\vec{c}$ in normal form. Then for $I := \{1 \leq i \leq n \mid a_i < b_i\}$,*

$$\vec{x} \geq |I|\vec{c} \quad \text{and} \quad S_{\vec{y}} \cdot S_{\vec{x} - \vec{y}} = \left(\prod_{i \in I} x_i^{p_i} \right) S_{\vec{x} - |I|\vec{c}}.$$

- (2) *Let X, Y be a basis of $S_{\vec{c}}$. If $\vec{x} \geq i\vec{c} \geq 0$, then*

$$S_{\vec{x}} = X S_{\vec{x} - \vec{c}} + f(X, Y) S_{\vec{x} - i\vec{c}}$$

for any $f(X, Y) \in S_{i\vec{c}}$ which is not a multiple of X .

Before proving the main result 4.17, we first illustrate a special case.

Example 4.16. Let $\vec{s}_a = \sum_{i=1}^n \vec{x}_i + a\vec{c}$ with $a \geq 0$ and $n + a \geq 2$ (since $a \geq 0$, the last condition is equivalent to $\vec{s}_a \notin [0, \vec{c}]$). Then $S(\vec{y})^{\vec{s}_a}$ is a special CM $S^{\vec{s}_a}$ -module for all $\vec{y} \in [0, \vec{c}]$.

Proof. We use 4.14. When $\ell \leq 0$, both sides of (4.H) are zero. When $\ell > 0$, since $\vec{\omega} + \vec{s}_a = (n - 2 + a)\vec{c}$ we have

$$S_{\vec{y} + \vec{\omega} + \ell\vec{s}_a} = S_{\vec{y} + (\ell-1)\vec{s}_a + (n-2+a)\vec{c}} \stackrel{4.10(1)}{=} S_{(n-2+a)\vec{c}} \cdot S_{\vec{y} + (\ell-1)\vec{s}_a} = S_{\vec{\omega} + \vec{s}_a} \cdot S_{\vec{y} + (\ell-1)\vec{s}_a}$$

and so (4.H) holds. \square

The following is the main result in this section, and does not assume any of the geometry from §3. The geometry is required to obtain the \supseteq statement in part (2), as the algebra only gives the inclusion \subseteq . However, the algebraic method of proof developed below feeds back into the geometry, and allows us to extract the middle self-intersection number in 4.21. As notation, we denote those special CM R -modules that are \mathbb{Z} -graded by $\mathbf{SCM}^{\mathbb{Z}}R$.

Theorem 4.17. *Let $\vec{x} \in \mathbb{L}_+$ with $\vec{x} \notin [0, \vec{c}]$ and $R := S^{\vec{x}}$. Write $\vec{x} = \sum_{i=1}^n a_i \vec{x}_i + a\vec{c}$ in normal form, then the following statements hold.*

- (1) *Up to degree shift, the objects of rank one in $\mathbf{SCM}^{\mathbb{Z}}R$ are precisely $S(u\vec{x}_j)^{\vec{x}}$ for $1 \leq j \leq n$ and u appearing in the i -series for $\frac{1}{p_j}(1, -a_j)$.*
(2) *Forgetting the grading, $\text{add}\{S(u\vec{x}_j)^{\vec{x}} \mid j \in [1, n], u \in I(p_j, p_j - a_j)\} \subseteq \mathbf{SCM} R$.*

In particular, R , $S(\vec{c})^{\vec{x}}$ and $S((p_j - a_j)\vec{x}_j)^{\vec{x}}$ for all $j \in [1, n]$ are always special.

Proof. We only prove (1), since the other statements follow immediately. By 4.12(1) we can assume that $(a_i, p_i) = 1$ for all $1 \leq i \leq n$.

(a) We first claim that, up to degree shift, \mathbb{Z} -graded special CM R -modules of rank one must have the form $S(u\vec{x}_j)^{\vec{x}}$ for some $1 \leq j \leq n$ and $0 \leq u \leq p_j$. By [GL1, 1.3] S is an \mathbb{L} -graded factorial domain, so all rank one objects in $\mathbf{CM}^{\mathbb{L}}S$ have the form $S(\vec{y})$ for some $\vec{y} \in \mathbb{L}$. Under the rank preserving equivalence 4.8(1), it follows that all rank one graded CM R -modules have the form $S(\vec{y})^{\vec{x}}$ for some $\vec{y} \in \mathbb{L}$. Since we are working up to degree shift, and $\vec{x} \geq 0$, we can assume without loss of generality that $\vec{y} \geq 0$ and $\vec{y} \not\geq \vec{x}$, by if necessary replacing \vec{y} by $\vec{y} - \ell\vec{x}$ for some $\ell \in \mathbb{Z}$.

Hence we can assume that our rank one special CM module has the form $S(\vec{y})^{\vec{x}}$ with $\vec{y} \geq 0$ and $\vec{y} \not\geq \vec{x}$. Now assume that \vec{y} can not be written as $u\vec{x}_j$ for some $1 \leq j \leq n$ and $0 \leq u \leq p_j$. Then there exists $j \neq k$ such that $\vec{y} \geq \vec{x}_j + \vec{x}_k$. By applying 4.14 for $\ell = 0$, it follows that

$$S_{\vec{y}+\vec{\omega}} = \sum_{m \in \mathbb{Z}} S_{\vec{\omega}+m\vec{x}} \cdot S_{\vec{y}-m\vec{x}}.$$

Now $S_{\vec{y}+\vec{\omega}} \neq 0$ by our assumption $\vec{y} \geq \vec{x}_j + \vec{x}_k$, hence there exists $m \in \mathbb{Z}$ such that $S_{\vec{\omega}+m\vec{x}} \neq 0$ and $S_{\vec{y}-m\vec{x}} \neq 0$. On one hand, since $\vec{\omega} \not\geq 0$, this implies that $m > 0$. On the other hand, since $\vec{y} \not\geq \vec{x}$, this implies that $m \leq 0$, a contradiction. Thus the rank 1 special CM modules have the claimed form.

(b) Let $1 \leq j \leq n$ and $0 \leq u \leq p_j$. We now show that $S(u\vec{x}_j)^{\vec{x}}$ is a special CM R -module if and only if u appears in the i -series for $\frac{1}{p_j}(1, -a_j)$. By 4.14, the CM R -module $S(u\vec{x}_j)^{\vec{x}}$ is special if and only if

$$S_{u\vec{x}_j+\vec{\omega}+\ell\vec{x}} = \sum_{m \in \mathbb{Z}} S_{\vec{\omega}+m\vec{x}} \cdot S_{u\vec{x}_j+(\ell-m)\vec{x}} \quad (4.K)$$

holds for all $\ell \in \mathbb{Z}$, or equivalently, for all $\ell > 0$ since the left hand side vanishes for $\ell \leq 0$ by $u\vec{x}_j + \vec{\omega} \leq \vec{c} + \vec{\omega} \not\geq 0$. Thus in what follows, we fix an arbitrary $\ell > 0$.

Note first that \supseteq in (4.K) is clear since the weight of each product in the right hand side is $u\vec{x}_j + \vec{\omega} + \ell\vec{x}$. Hence equality holds in (4.K) if and only if \subseteq holds. Also note that the term $S_{\vec{\omega}+m\vec{x}} \cdot S_{u\vec{x}_j+(\ell-m)\vec{x}}$ in the right hand side of (4.K) is non-zero only if $1 \leq m \leq \ell$ since $\vec{\omega} \not\geq 0$ and $u\vec{x}_j - \vec{x} \leq \vec{c} - \vec{x} \not\geq 0$; here we have used the assumption that $\vec{x} \notin [0, \vec{c}]$. Thus, to simplify notation, for $1 \leq m \leq \ell$ write

$$\begin{aligned} \vec{x} &:= u\vec{x}_j + \vec{\omega} + \ell\vec{x} \\ \vec{y}_m &:= \vec{\omega} + m\vec{x} \end{aligned}$$

then (4.K) holds if and only if

$$S_{\vec{x}} \subseteq \sum_{m=1}^{\ell} S_{\vec{y}_m} \cdot S_{\vec{x}-\vec{y}_m} \quad (4.L)$$

holds. Note that \vec{x} and \vec{y}_m can be written more explicitly as

$$\begin{aligned} \vec{x} &= \left(\sum_{i \neq j} (\ell a_i - 1) \vec{x}_i \right) + (u + \ell a_j - 1) \vec{x}_j + (n - 2 + \ell) \vec{c} \\ \vec{y}_m &= \sum_{i=1}^n (m a_i - 1) \vec{x}_i + (n - 2 + m) \vec{c}. \end{aligned} \quad (4.M)$$

Now by 4.15(1), for each $1 \leq m \leq \ell$ we have

$$S_{\vec{y}_m} \cdot S_{\vec{x}-\vec{y}_m} = \left(\prod_{i \in I_m} x_i^{p_i} \right) S_{\vec{x}-|I_m|\vec{c}} \quad (4.N)$$

where I_m is the set in 4.15(1). As before, for an integer k , we denote by $[k]_{p_i}$ the integer k' satisfying $0 \leq k' \leq p_i - 1$ and $k - k' \in p_i \mathbb{Z}$. Simply writing out \vec{x} and \vec{y}_m into normal form, from (4.M) we see that

$$I_m = \{1 \leq i \leq n \mid [u_i + \ell a_i - 1]_{p_i} < [m a_i - 1]_{p_i}\} \quad (4.O)$$

where $u_i := u$ if $i = j$ and $u_i := 0$ otherwise.

For the case $m = \ell$, it is clear that $I_\ell \subseteq \{j\}$. Hence we see that

$$S_{\vec{y}_\ell} \cdot S_{\vec{x}-\vec{y}_\ell} \stackrel{(4.N)}{=} \left(\prod_{i \in I_\ell} x_i^{p_i} \right) S_{\vec{x}-|I_\ell|\vec{c}} \supseteq x_j^{p_j} S_{\vec{x}-\vec{c}}. \quad (4.P)$$

Now we claim that (4.L) holds if and only if $j \notin I_m$ for some $1 \leq m \leq \ell$.

(\Rightarrow) Assume that (4.L) holds. If further $j \in I_m$ for all $1 \leq m \leq \ell$, then using

$$S_{\vec{x}} \stackrel{(4.L)}{=} \sum_{m=1}^{\ell} S_{\vec{y}_m} \cdot S_{\vec{x}-\vec{y}_m} \stackrel{(4.N)}{=} \sum_{m=1}^{\ell} \left(\prod_{i \in I_m} x_i^{p_i} \right) S_{\vec{x}-|I_m|\vec{c}}$$

we see that $x_j^{p_j}$ divides every element in $S_{\vec{x}}$. This gives a contradiction, since we can use the normal form of \vec{x} to obtain elements of $S_{\vec{x}}$ which are not divisible by $x_j^{p_j}$.

(\Leftarrow) Suppose that $j \notin I_m$ for some $1 \leq m \leq \ell$. Since $\vec{x} \geq |I_m|\vec{c} \geq 0$ holds by 4.15(1), we have

$$S_{\vec{x}} = x_j^{p_j} S_{\vec{x}-\vec{c}} + \left(\prod_{i \in I_m} x_i^{p_i} \right) S_{\vec{x}-|I_m|\vec{c}}.$$

by choosing $X := x_j^{p_j}$ and $f(X, Y) := \prod_{i \in I_m} x_i^{p_i}$ in 4.15(2). Finally, using (4.N) and (4.P) this gives

$$S_{\vec{x}} \subseteq S_{\vec{y}_\ell} \cdot S_{\vec{x}-\vec{y}_\ell} + S_{\vec{y}_m} \cdot S_{\vec{x}-\vec{y}_m},$$

which clearly implies (4.L).

Consequently, (4.L) holds if and only if $j \notin I_m$ for some $1 \leq m \leq \ell$, which by (4.O) holds if and only if $[u + \ell a_j - 1]_{p_j} \geq [m a_j - 1]_{p_j}$ for some $1 \leq m \leq \ell$. By 2.18, this holds if and only if u appears in the i -series for $\frac{1}{p_j}(1, -a_j)$. \square

4.5. The Middle Self-Intersection Number. In this subsection we use the techniques of the previous subsections to determine the middle self-intersection number in (1.E). This requires the following two elementary but technical lemmas.

Lemma 4.18. *Let ℓ_1, \dots, ℓ_m be elements in $S_{\vec{c}}$ such that any two elements are linearly independent. Then $\prod_{j \neq 1} \ell_j, \dots, \prod_{j \neq m} \ell_j$ is a basis of $S_{(m-1)\vec{c}}$.*

Proof. Assume that the assertion holds for $m-1$. Then $\prod_{j \neq 1} \ell_j, \dots, \prod_{j \neq m-1} \ell_j$ gives a basis of $\ell_m S_{(m-2)\vec{c}}$. Since $S_{(m-1)\vec{c}} = \ell_m S_{(m-2)\vec{c}} + k \prod_{j \neq m} \ell_j$ holds, the assertion also holds for m . \square

The following lemma is general, and does not require $n > 0$.

Lemma 4.19. *Let $\vec{x} \in \mathbb{L}_+$, and write $\vec{x} = \sum_{i=1}^n a_i \vec{x}_i + a \vec{c}$ in normal form. If $t \geq 2$, then every morphism in $\text{Hom}_S^{\mathbb{L}}(S(\vec{c}), S(t\vec{x}))$ factors through $\text{add}\{S(\vec{x} + (p_i - a_i)\vec{x}_i) \mid 1 \leq i \leq n\}$.*

Proof. It suffices to show that

$$S_{t\vec{x}-\vec{c}} \subset \sum_{i=1}^n S_{\vec{x}-a_i \vec{x}_i} \cdot S_{(t-1)\vec{x}-(p_i-a_i)\vec{x}_i}.$$

For each i with $1 \leq i \leq n$, take $m_i \geq 0$ and $\varepsilon_i \in \{0, 1\}$ such that

$$(t-1)a_i = [(t-1)a_i]_{p_i} + m_i p_i \quad \text{and} \quad t a_i = [t a_i]_{p_i} + (m_i + \varepsilon_i) p_i.$$

Let $m := \sum_{i=1}^n m_i$ and $\varepsilon := \sum_{i=1}^n \varepsilon_i$. Then the equality

$$t\vec{x} - \vec{c} = \sum_{j=1}^n [t a_j]_{p_j} \vec{x}_j + (m + \varepsilon - 1 + t a) \vec{c}$$

implies that

$$S_{t\vec{x}-\vec{c}} = \left(\prod_{j=1}^n x_j^{[t a_j]_{p_j}} \right) S_{(m+\varepsilon-1+ta)\vec{c}}. \quad (4.Q)$$

Similarly the equality

$$(t-1)\vec{x} - (p_i - a_i)\vec{x}_i = [t a_i]_{p_i} \vec{x}_i + \sum_{j \neq i} [(t-1)a_j]_{p_j} \vec{x}_j + (m + \varepsilon_i - 1 + (t-1)a)\vec{c}$$

implies that

$$S_{(t-1)\vec{x}-(p_i-a_i)\vec{x}_i} = x_i^{[t a_i]_{p_i}} \left(\prod_{j \neq i} x_j^{[(t-1)a_j]_{p_j}} \right) S_{(m+\varepsilon_i-1+(t-1)a)\vec{c}}.$$

Multiplying $S_{\vec{x}-a_i \vec{x}_i} = \left(\prod_{j \neq i} x_j^{a_j} \right) S_{a\vec{c}}$ and using $[(t-1)a_j]_{p_j} + a_j = [t a_j]_{p_j} + \varepsilon_j p_j$ gives

$$S_{\vec{x}-a_i \vec{x}_i} \cdot S_{(t-1)\vec{x}-(p_i-a_i)\vec{x}_i} = \left(\prod_{j=1}^n x_j^{[t a_j]_{p_j}} \right) \left(\prod_{j \neq i} x_j^{\varepsilon_j p_j} \right) S_{a\vec{c}} \cdot S_{(m+\varepsilon_i-1+(t-1)a)\vec{c}}. \quad (4.R)$$

Now set $I := \{1 \leq i \leq n \mid \varepsilon_i = 1\}$. Clearly $|I| = \varepsilon$ holds.

First we assume $I \neq \emptyset$. By 4.18 we have $\sum_{i \in I} k \prod_{j \neq i} x_j^{\varepsilon_j p_j} = S_{(\varepsilon-1)\vec{c}}$ and thus

$$\begin{aligned} \sum_{i \in I} S_{\vec{x}-a_i \vec{x}_i} \cdot S_{(t-1)\vec{x}-(p_i-a_i)\vec{x}_i} &\stackrel{(4.R)}{=} \left(\prod_{j=1}^n x_j^{[ta_j]p_j} \right) S_{(\varepsilon-1)\vec{c}} \cdot S_{a\vec{c}} \cdot S_{(m+(t-1)a)\vec{c}} \\ &= \left(\prod_{j=1}^n x_j^{[ta_j]p_j} \right) S_{(m+\varepsilon-1+ta)\vec{c}} \\ &\stackrel{(4.Q)}{=} S_{t\vec{x}-\vec{c}}, \end{aligned}$$

as desired.

Next we assume $I = \emptyset$. If further $m-1+(t-1)a \geq 0$, then (4.R) is equal to

$$\left(\prod_{j=1}^n x_j^{[ta_j]p_j} \right) S_{a\vec{c}} \cdot S_{(m-1+(t-1)a)\vec{c}} = \left(\prod_{j=1}^n x_j^{[ta_j]p_j} \right) S_{(m-1+ta)\vec{c}} \stackrel{(4.Q)}{=} S_{t\vec{x}-\vec{c}},$$

as desired, so we can assume that $m-1+(t-1)a < 0$. But $a \geq 0$ since $\vec{x} \in \mathbb{L}_+$, and $t \geq 2$ by assumption, so necessarily $m=0=a$. Then $m-1+ta < 0$ holds, so $S_{t\vec{x}-\vec{c}}=0$ by (4.Q), which implies the assertion. \square

The following is the main algebraic result of this subsection; the main point is that the manipulations above involving the combinatorics of the weighted projective line give the geometric corollary in 4.21 below.

Theorem 4.20. *Let $\vec{x} \in \mathbb{L}_+$ with $\vec{x} \notin [0, \vec{c}]$, and write $\vec{x} = \sum_{i=1}^n a_i \vec{x}_i + a\vec{c}$ in normal form. Set $R := S^{\vec{x}}$ and $N := S(\vec{c})^{\vec{x}}$, and consider their completions \mathfrak{R} and \hat{N} . Then in the quiver of the reconstruction algebra of \mathfrak{R} , the number of arrows from \hat{N} to \mathfrak{R} is a .*

Proof. By 4.12(2), $S_{\vec{p}, \lambda}^{\vec{x}} \cong S_{\vec{p}', \lambda'}^{\vec{x}}$ as \mathbb{Z} -graded algebras, where $\vec{x} := \sum_{i \in I} a_i \vec{x}_i + a\vec{c} \in \mathbb{L}$ satisfies the hypothesis in 4.8(4). Note that this change in parameters has not changed the value a on \vec{c} , hence in what follows, we can assume that $\mathbf{CM}^{\mathbb{L}} S \simeq \mathbf{CM}^{\mathbb{Z}} R$ holds, via the functor $(-)^{\vec{x}}$.

Let \mathcal{C} be the full subcategory of $\mathbf{CM}^{\mathbb{L}} S$ corresponding to $\mathbf{SCM}^{\mathbb{Z}} R$ via the functor $(-)^{\vec{x}}$. Then the number of arrows from \hat{N} to \mathfrak{R} is equal to the dimension of the k -vector space

$$\frac{\text{rad}_{\mathbf{SCM}^{\mathbb{Z}} \mathfrak{R}}^2(\hat{N}, \mathfrak{R})}{\text{rad}_{\mathbf{SCM}^{\mathbb{Z}} \mathfrak{R}}^2(\hat{N}, \mathfrak{R})} \cong \prod_{t \in \mathbb{Z}} \frac{\text{Hom}_{\mathbb{Z}}^{\mathbb{Z}}(N, R(t))}{\text{rad}_{\mathbf{SCM}^{\mathbb{Z}} R}^2(N, R(t))} \cong \prod_{t \in \mathbb{Z}} \frac{\text{Hom}_{\mathbb{L}}^{\mathbb{L}}(S(\vec{c}), S(t\vec{x}))}{\text{rad}_{\mathcal{C}}^2(S(\vec{c}), S(t\vec{x}))}.$$

By 4.17, \mathcal{C} is the additive closure of $S(u\vec{x}_j + s\vec{x})$, where $s \in \mathbb{Z}$, $1 \leq j \leq n$ and u appears in the i -series for $\frac{1}{p_j}(1, -a_j)$. We split into three cases.

- (1) If $t \leq 0$, then $\text{Hom}_{\mathbb{L}}^{\mathbb{L}}(S(\vec{c}), S(t\vec{x})) = 0$.
- (2) If $t \geq 2$, then since $S(\vec{x} + (p_i - a_i)\vec{x}_i)$ belongs to \mathcal{C} and is not isomorphic to both $S(\vec{c})$ and $S(t\vec{x})$ (since $\vec{x} \notin [0, \vec{c}]$), we have $\text{Hom}_{\mathbb{L}}^{\mathbb{L}}(S(\vec{c}), S(t\vec{x})) = \text{rad}_{\mathcal{C}}^2(S(\vec{c}), S(t\vec{x}))$ by 4.19.
- (3) Suppose that $t = 1$. By definition any morphism in $\text{rad}_{\mathcal{C}}^2(S(\vec{c}), S(\vec{x}))$ can be written as a sum of compositions $S(\vec{c}) \rightarrow S(u\vec{x}_j + s\vec{x}) \rightarrow S(\vec{x})$. If $s \leq 0$, then

$$\text{Hom}_{\mathbb{L}}^{\mathbb{L}}(S(\vec{c}), S(u\vec{x}_j + s\vec{x})) \cong S_{u\vec{x}_j + s\vec{x} - \vec{c}} = 0,$$

and if $s \geq 1$, then

$$\text{Hom}_{\mathbb{L}}^{\mathbb{L}}(S(u\vec{x}_j + s\vec{x}), S(\vec{x})) \cong S_{(1-s)\vec{x} - u\vec{x}_j} = 0.$$

It follows that $\text{rad}_{\mathcal{C}}^2(S(\vec{c}), S(t\vec{x})) = 0$ in this case.

Combining all cases, the desired number is thus

$$\sum_{t \in \mathbb{Z}} \dim_k \left(\frac{\text{Hom}_{\mathbb{L}}^{\mathbb{L}}(S(\vec{c}), S(t\vec{x}))}{\text{rad}_{\mathcal{C}}^2(S(\vec{c}), S(t\vec{x}))} \right) = \dim_k \text{Hom}_{\mathbb{L}}^{\mathbb{L}}(S(\vec{c}), S(\vec{x})) = \dim_k S_{\vec{x}-\vec{c}} \stackrel{4.10}{=} a. \quad \square$$

This allows us to finally complete the proof of 1.2 from the introduction.

Corollary 4.21. *Let $\vec{x} \in \mathbb{L}_+$ with $\vec{x} \notin [0, \vec{c}]$, and write $\vec{x} = \sum_{i=1}^n a_i \vec{x}_i + a\vec{c}$ in normal form. Then $\pi: Y^{\vec{x}} \rightarrow \text{Spec } S^{\vec{x}}$ is the minimal resolution, and its dual graph is precisely (1.E) with $\beta = a + v = a + \#\{i \mid a_i \neq 0\}$.*

Proof. We know from 3.9 that π is the minimal resolution, and we know from construction of $Y^{\vec{x}}$ that all the self-intersection numbers are determined by the continued fraction expansions, except the middle curve E_i corresponding to the special CM module $S(\vec{c})^{\vec{x}}$. The dual graph does not change under completion. By 4.20 the number of arrows in the reconstruction algebra from the middle vertex to the vertex \circ is a . Thus the calculation (2.B) combined with 2.9 shows that $a = -E_i \cdot Z_f = \beta - v$. \square

4.6. The Reconstruction Algebra and its qgr. Using the above subsections, we next describe the quiver of the reconstruction algebra and determine the associated **qgr** category. Consider the dual graph (1.E), then with the convention that we only draw the arms that are non-empty, we see from (2.B) and $Z_K \cdot E_i = E_i^2 + 2$ that

$$((Z_K - Z_f) \cdot E_i)_i = \begin{array}{ccccccc} & 1 & & 1 & & 1 & & 1 \\ & | & & | & & | & & | \\ 0 & & 0 & & 0 & & \cdots & 0 \\ \vdots & & \vdots & & \vdots & & & \vdots \\ 0 & & 0 & & 0 & & \cdots & 0 \\ & | & & | & & | & & | \\ & 0 & & 0 & & 0 & & 0 \end{array} \quad (4.S)$$

$\swarrow \quad \searrow \quad \downarrow \quad \swarrow \quad \searrow$
 $2-v$

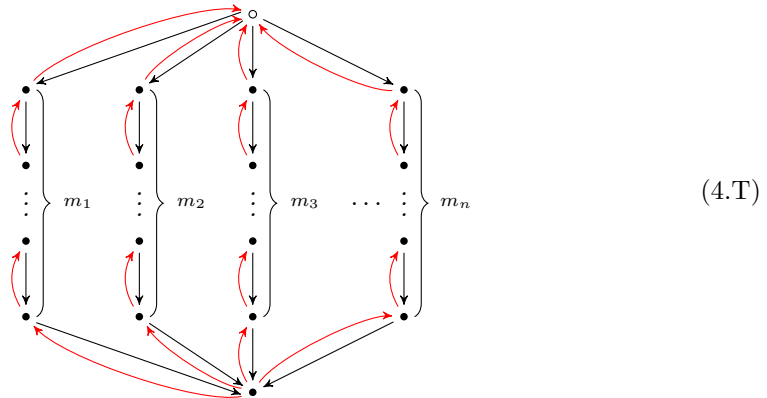
Note that the cases $v = 0$ and $v = 1$ are degenerate, and are already well understood [W3]. Therefore in the next result, we only consider the case $v \geq 2$.

Inspecting the list of special CM $S^{\vec{x}}$ -modules in 3.10, the conditions in 2.8 are satisfied, so we consider the particular choice of reconstruction algebra

$$\Gamma_{\vec{x}} := \text{End}_{S^{\vec{x}}} \left(S^{\vec{x}} \oplus \left(\bigoplus_{j \in [1, n], u} S(u\vec{x}_j)^{\vec{x}} \right) \oplus S^{\vec{x}}(\vec{c}) \right),$$

where u in the middle direct sum ranges over $I(p_j, p_j - a_j) \setminus \{0, p_j\}$. Since the above $S^{\vec{x}}$ -modules are clearly \mathbb{Z} -graded, this induces a \mathbb{Z} grading on $\Gamma_{\vec{x}}$.

Corollary 4.22. *For $\vec{x} \in \mathbb{L}_+$ with $\vec{x} \notin [0, \vec{c}]$, write $\vec{x} = \sum_{i=1}^n a_i \vec{x}_i + a\vec{c}$ in normal form. For each i with $a_i \neq 0$, as before m_i is defined via $\frac{p_i}{p_i - a_i} = [\alpha_{i1}, \dots, \alpha_{im_i}]$, and if $a_i = 0$ set $m_i = 0$. Suppose that $v = \#\{i \mid a_i \neq 0\}$ satisfies $v \geq 2$. Then the reconstruction algebra $\Gamma_{\vec{x}}$ can be presented as a quiver with relations, where the relations are homogeneous with respect to the natural grading, and the quiver is the following: we first consider the double quiver of the dual graph (1.E) and add an extending vertex (denoted \circ) as follows:*



where by convention if $m_i = 0$ the i th arm does not exist. Further, we add extra arrows subject to the following rules:

- (1) If some $\alpha_{ij} > 2$, add $\alpha_{ij} - 2$ extra arrows from that vertex to the top vertex.
- (2) Add further a arrows from the bottom vertex to the top vertex.

Proof. As in [W4, §4], we first work on the completion, which is naturally filtered, then go back to the graded setting by taking the associated graded ring. Doing this, the result is then immediate from (2.B), (4.S) and 2.9. \square

It is possible to describe the relations too in this level of generality, but for notational ease we will only do this for the 0-Wahl Veronese in §5 below. However, in full generality, we do have the following:

Proposition 4.23. *Suppose that $0 \neq \vec{x} = \sum_{i=1}^n a_i \vec{x}_i + a\vec{c} \in \mathbb{L}_+$ with $\vec{x} \notin [0, \vec{c}]$. Then, with notation as in 4.22, $(\Gamma_{\vec{x}})_0$, the degree zero part of the reconstruction algebra $\Gamma_{\vec{x}}$, is isomorphic to the canonical algebra $\Lambda_{\mathbf{q}, \mu}$, where $I := \{i \in [1, n] \mid a_i \neq 0\}$, $\mathbf{q} := (m_i + 1)_{i \in I}$ and $\mu := (\lambda_i)_{i \in I}$.*

Proof. By 4.12 we can change parameters to assume that the coprime assumption of 4.8 holds. Thus we have $\mathrm{CM}^{\mathbb{Z}} R \simeq \mathrm{CM}^{\mathbb{L}} S$ and hence

$$(\Gamma_{\vec{x}})_0 \cong \mathrm{End}_S^{\mathbb{L}}\left(\bigoplus_u S(u\vec{x}_i)\right),$$

where u ranges over the respective i -series. But by [GL1] it is well known that the ring $\mathrm{End}_S^{\mathbb{L}}(\bigoplus_{\vec{y} \in [0, \vec{c}]} S(\vec{y})^{\vec{x}})$ is isomorphic to a canonical algebra, and thus $(\Gamma_{\vec{x}})_0$ is obtained from this by composing arrows through the vertices which do not appear in the i -series. It is clear that this gives the canonical algebra $\Lambda_{\mathbf{q}, \mu}$. \square

When $\vec{x} \in \mathbb{L}_+$ with $\vec{x} \notin [0, \vec{c}]$, we will next show in 4.25 that $\mathrm{qgr}^{\mathbb{Z}} S^{\vec{x}} \simeq \mathrm{qgr}^{\mathbb{Z}} \Gamma_{\vec{x}}$, since this then allows us to interpret the weighted projective line as a ‘noncommutative scheme’ over the canonical algebra. This result requires two lemmas. As notation, set $R := S^{\vec{x}}$, let $N := \bigoplus_u S(u\vec{x}_i)^{\vec{x}}$ be the sum of all the indecomposable special CM R -modules.

Lemma 4.24. $\mathrm{End}_{\mathrm{qgr}^{\mathbb{Z}} R}(N) \cong \mathrm{End}_{\mathrm{gr}^{\mathbb{Z}} R}(N) \cong (\Gamma_{\vec{x}})_0 \cong \Lambda_{\mathbf{q}, \mu}$.

Proof. Since $N \in \mathrm{mod}_2^{\mathbb{Z}} R$ the first isomorphism is [GL2, 2.1]. The second one is clear. The final isomorphism is 4.23. \square

Proposition 4.25. *For $\vec{x} \in \mathbb{L}_+$ with $\vec{x} \notin [0, \vec{c}]$, there is an equivalence*

$$\mathrm{qgr}^{\mathbb{Z}} S^{\vec{x}} \simeq \mathrm{qgr}^{\mathbb{Z}} \Gamma_{\vec{x}},$$

where $\Gamma_{\vec{x}}$ is a \mathbb{Z} -graded k -algebra such that $(\Gamma_{\vec{x}})_i$ is $\Lambda_{\mathbf{q}, \mu}$ for $i = 0$ and zero for $i < 0$.

Proof. We apply 4.1, with the last statement simply being 4.24. Let $B := \Gamma = \mathrm{End}_A(N)$ and let e be the idempotent corresponding to the summand R . Clearly $A := eBe \cong R$. Note that $\dim_k(B/\langle e \rangle) < \infty$ since the normality of R implies that $\mathrm{add} N_{\mathfrak{p}} = \mathrm{add} R_{\mathfrak{p}}$ for any non-maximal ideal \mathfrak{p} of R . This shows that $\mathrm{End}_{R_{\mathfrak{p}}}(N_{\mathfrak{p}})$ is a matrix algebra over $R_{\mathfrak{p}}$ for all non-maximal \mathfrak{p} , so $(B/\langle e \rangle)_{\mathfrak{p}} = 0$. \square

5. THE 0-WAHL VERONESE

Throughout this section we work with an arbitrary $\mathbb{X}_{\mathbf{p}, \lambda}$ with $n \geq 3$, and consider the 0-Wahl Veronese from the introduction, namely $S^{\vec{s}}$, where $\vec{s} = \sum_{i=1}^n \vec{x}_i$. It is not too hard, but more notationally complicated, to extend to cover the case $\vec{s}_a = \vec{s} + a\vec{c}$, but we shall not do this here. We investigate the more general $S^{\vec{s}_a}$ for Dynkin type in §6.

5.1. Presenting the 0-Wahl Veronese. The aim of this subsection is to give a presentation of the 0-Wahl Veronese subring $S^{\vec{s}}$ of S by constructing an isomorphism $S^{\vec{s}} \cong R_{\mathbf{p}, \lambda}$. We define elements of $S^{\vec{s}}$ as follows:

$$\begin{aligned} \mathbf{u}_i &:= \begin{cases} x_1^{p_1+p_2} x_3^{p_2} \dots x_n^{p_2} & i = 1, \\ x_2^{p_1+p_2} x_3^{p_1} \dots x_n^{p_1} & i = 2, \\ -x_1^{p_i} x_2^{p_2+p_i} x_3^{p_i} \dots \widehat{x_i} \dots x_n^{p_i} & 3 \leq i \leq n, \end{cases} \\ \mathbf{v} &:= x_1 x_2 \dots x_n, \end{aligned}$$

where we write $\widehat{x_i}$ to mean ‘omit x_i ’. Then \mathbf{v} is homogeneous of degree one, and \mathbf{u}_i is homogeneous of degree p_2 if $i = 1$, p_1 if $i = 2$ and p_i if $3 \leq i \leq n$.

To construct an isomorphism between $R_{\mathbf{p}, \lambda}$ and $S^{\vec{s}}$, we first construct a morphism of graded algebras.

Lemma 5.1. *The morphism $k[u_1, \dots, u_n, v] \rightarrow S^{\vec{s}}$ of graded algebras given by $u_i \mapsto \mathbf{u}_i$ for $1 \leq i \leq n$ and $v \mapsto \mathbf{v}$ induces a morphism $R_{\mathbf{p}, \lambda} \rightarrow S^{\vec{s}}$ of graded algebras.*

Proof. It suffices to show that all 2×2 minors of the following matrix has determinant zero.

$$\begin{pmatrix} \mathbf{u}_2 & \mathbf{u}_3 & \dots & \mathbf{u}_n & \mathbf{v}^{p_2} \\ \mathbf{v}^{p_1} & \lambda_3 \mathbf{u}_3 + \mathbf{v}^{p_3} & \dots & \lambda_n \mathbf{u}_n + \mathbf{v}^{p_n} & \mathbf{u}_1 \end{pmatrix} \quad (5.A)$$

Since $S^{\vec{s}}$ is a domain, it suffices to show that all 2×2 minors containing the last column has determinant zero. The outer 2×2 minor has determinant

$$\mathbf{u}_1 \mathbf{u}_2 - \mathbf{v}^{p_1+p_2} = (x_1 \dots x_n)^{p_1+p_2} - (x_1 \dots x_n)^{p_1+p_2} = 0.$$

Further for any $i \geq 3$, using the relation $x_1^{p_1} = \lambda_i x_2^{p_2} - x_i^{p_i}$ it follows that

$$\begin{aligned} \mathbf{u}_1 \mathbf{u}_i &= -x_1^{p_1+p_2+p_i} x_2^{p_2+p_i} x_3^{p_3+p_i} \dots x_i^{p_i} \dots x_n^{p_n+p_i} \\ &= -(\lambda_i x_2^{p_2} - x_i^{p_i}) x_1^{p_2+p_i} x_2^{p_2+p_i} x_3^{p_3+p_i} \dots x_i^{p_i} \dots x_n^{p_n+p_i} \\ &= x_1^{p_2} x_2^{p_2} \dots x_n^{p_n} (-\lambda_i x_1^{p_i} x_2^{p_2+p_i} x_3^{p_3} \dots \widehat{x_i} \dots x_n^{p_i} + x_1^{p_i} x_2^{p_i} \dots x_n^{p_i}) \\ &= \mathbf{v}^{p_2} (\lambda_i \mathbf{u}_i + \mathbf{v}^{p_i}). \end{aligned}$$

Thus the 2×2 minor consisting of i th column and the last one has determinant zero. \square

The following calculation is elementary.

Proposition 5.2. (1) *The k -algebra $S^{\vec{s}}$ is generated by \mathbf{v} and \mathbf{u}_i with $1 \leq i \leq n$.*
 (2) *The k -vector space $S^{\vec{s}}/\mathbf{v}S^{\vec{s}}$ is generated by \mathbf{u}_i^ℓ with $1 \leq i \leq n$ and $\ell \geq 0$.*

Proof. It is enough to prove (2). Let V be a subspace of $S^{\vec{s}}/\mathbf{v}S^{\vec{s}}$ generated by \mathbf{u}_i^ℓ with $1 \leq i \leq n$ and $\ell \geq 0$. Take any monomial $X := x_1^{a_1} \dots x_n^{a_n}$ in $S_{N\vec{s}}$ with $N > 0$, then

$$a_1 \vec{x}_1 + \dots + a_n \vec{x}_n = N \vec{x}_1 + \dots + N \vec{x}_n.$$

For each $1 \leq i \leq n$, there exists $\ell_i \in \mathbb{Z}$ such that $a_i = N + \ell_i p_i$. Then $\sum_{i=1}^n \ell_i = 0$ holds.

(i) We show that X belongs to V if there exists $1 \leq i \leq n$ satisfying $\ell_i \leq 0$, $\ell_j \geq 0$ and $\ell_k = 0$ for all $k \neq i, j$, where j is defined by $j := 2$ if $i \neq 2$ and $j := 1$ if $i = 2$.

If $a_i \neq 0$, then X belongs to $\mathbf{v}S^{\vec{s}}$. Assume $a_i = 0$. Then $N = -\ell_i p_i$ and $a_j = N + \ell_j p_j = -\ell_i(p_i + p_j)$ hold. We have

$$X = x_j^{a_j} \prod_{k \neq i, j} x_k^{N} = (x_j^{p_i+p_j} \prod_{k \neq i, j} x_k^{p_i})^{-\ell_i} = \pm \mathbf{u}_{i'}^{-\ell_i},$$

where $i' := 2$ if $i = 1$, $i' := 1$ if $i = 2$ and $i' = i$ if $i \geq 3$. Thus the assertion follows.

(ii) We consider the general case. Using induction on $\ell(X) := \sum_{1 \leq i \leq n, \ell_i > 0} \ell_i$, we show that X belongs to V .

Assume $\ell(X) = 0$. Then $X = \mathbf{v}^N$ holds, and hence X belongs to V .

Assume that there exist $1 \leq i \neq j \leq n$ such that $\ell_i < 0$ and $\ell_j < 0$. Take $1 \leq k \leq n$ such that $\ell_k > 0$. Using the relation $x_k^{p_k} = \lambda' x_i^{p_i} + \lambda'' x_j^{p_j}$ with $\lambda', \lambda'' \in k$, we have $X = \lambda' X' + \lambda'' X''$ for some monomials X', X'' satisfying $\ell(X') < \ell(X)$ and $\ell(X'') < \ell(X)$. Since X' and X'' belongs to V , so does X .

In the rest, assume that there exists unique $1 \leq i \leq n$ satisfying $\ell_i < 0$. Define j by $j := 2$ if $i \neq 2$ and $j := 1$ if $i = 2$. Using the relation $x_k^{p_k} = \lambda'_k x_i^{p_i} + \lambda''_k x_j^{p_j}$ with $\lambda'_k, \lambda''_k \in k$, we have

$$X = x_i^{a_i} x_j^{a_j} \prod_{k \neq i, j} x_k^N (\lambda'_k x_i^{p_i} + \lambda''_k x_j^{p_j})^{\ell_k}.$$

This is a linear combination of monomials $Y = x_i^{b_i} x_j^{b_j} \prod_{k \neq i, j} x_k^N$ which satisfies the condition in (i). Thus X belongs to V . \square

The above leads to the following, which is the main result of this subsection.

Theorem 5.3. *There is an isomorphism $R_{\mathbf{p}, \lambda} \cong S^{\vec{s}}$ of graded algebras given by $u_i \mapsto \mathbf{u}_i$ for $1 \leq i \leq n$ and $v \mapsto \mathbf{v}$.*

Proof. Combining 5.1 and 5.2, there is a surjective graded ring homomorphism $\vartheta: R_{\mathbf{p}, \lambda} \rightarrow S^{\vec{s}}$. But now $R_{\mathbf{p}, \lambda}$, being a rational surface singularity, is automatically a domain. Since $S^{\vec{s}}$ is two-dimensional, ϑ must be an isomorphism. \square

5.2. Special CM $S^{\vec{s}}$ -Modules and the Reconstruction Algebra. The benefit of our Veronese construction of $R_{\mathbf{p},\lambda}$ is that it also produces the special CM modules, and we now describe these explicitly as 2-generated ideals. We first do this in the notation of S , then translate into the co-ordinates u_1, \dots, u_n, v .

Proposition 5.4. *The following are, up to degree shift, precisely the \mathbb{Z} -graded special CM $S^{\vec{s}}$ -modules. Moreover, they have the following generators and degrees:*

Module	Generators	Degree of generators
$S(q\vec{x}_1)^{\vec{s}}$	$x_2^{p_2}(x_2x_3 \dots x_n)^{p_1-q}$ and x_1^q	$p_1 - q$ and 0
$S(q\vec{x}_2)^{\vec{s}}$	$x_1^{p_1}(x_1x_3 \dots x_n)^{p_2-q}$ and x_2^q	$p_2 - q$ and 0
$S(q\vec{x}_i)^{\vec{s}}$	$x_2^{p_2}(x_1 \dots \widehat{x_i} \dots x_n)^{p_i-q}$ and x_i^q	$p_i - q$ and 0
$S(\vec{c})^{\vec{s}}$	$x_1^{p_1}$ and $x_2^{p_2}$	0 and 0

where in row one $q \in [1, p_1]$, in row two $q \in [1, p_2]$, and in row three $i \in [3, n]$, $q \in [1, p_i]$.

Proof. The first statement is 4.17(1). We only prove the generators (and their degrees) for $S(q\vec{x}_1)^{\vec{s}}$ since all other cases are similar. Let M be a submodule of $S(q\vec{x}_1)^{\vec{s}}$ generated by $g_1 := x_1^q$ and $g_2 := x_2^{p_1+p_2-q}x_3^{p_1-q} \dots x_n^{p_1-q}$. To prove $M = S(q\vec{x}_1)^{\vec{s}}$, it suffices to show that any monomial $X = x_1^{a_1} \dots x_n^{a_n} \in S(q\vec{x}_1)^{\vec{s}}$ of degree $N \geq 0$ has either g_1 or g_2 as a factor. Since

$$a_1\vec{x}_1 + \dots + a_n\vec{x}_n = (N+q)\vec{x}_1 + N\vec{x}_2 + \dots + N\vec{x}_n.$$

holds, there exists $\ell_i \in \mathbb{Z}$ for each $1 \leq i \leq n$ such that $a_1 = N+q+\ell_1p_1$ and $a_i = N+\ell_ip_i$ for $i \geq 2$. Then $\sum_{i=1}^n \ell_i = 0$ holds.

- (i) If $a_1 \geq q$, then X belongs to M since X has $g_1 = x_1^q$ as a factor.
- (ii) We show that X belongs to M if $\ell_3 = \dots = \ell_n = 0$. By (i), we can assume that $a_1 < q$ and hence $\ell_1 < 0$. Then $N = a_1 - q - \ell_1p_1 \geq p_1 - q$ holds. Since $\ell_2 = -\ell_1 > 0$, we have $a_2 = N + \ell_2p_2 \geq p_1 + p_2 - q$, which implies that $X = x_1^{a_1}x_2^{a_2}x_3^N \dots x_n^N$ has $g_2 = x_2^{p_1+p_2-q}x_3^{p_1-q} \dots x_n^{p_1-q}$ as a factor.
- (iii) We show that X belongs to M if all ℓ_3, \dots, ℓ_n are non-positive. By (i), we can assume that $a_1 < q$ and hence $\ell_1 < 0$. Using $\ell_2 = -\sum_{i \neq 2} \ell_i$ and the relation $x_2^{p_2} = \lambda'_i x_1^{p_1} + \lambda''_i x_i^{p_i}$, it follows that

$$X = x_1^{a_1}x_2^{N-\sum_{i \neq 2} \ell_i p_2}x_3^{a_3} \dots x_n^{a_n} = x_1^{a_1}x_2^{N-\ell_1 p_2} \prod_{i \geq 3} x_i^{a_i} (\lambda'_i x_1^{p_1} + \lambda''_i x_i^{p_i})^{-\ell_i}.$$

This is a linear combination of monomials which have $g_1 = x_1^q$ as a factor and a monomial $x_1^{a_1}x_2^{N-\ell_1 p_2} \prod_{i \geq 3} x_i^{a_i - \ell_i p_i} = x_1^{a_1}x_2^{N-\ell_1 p_2}x_3^N \dots x_n^N$ satisfying (ii). Thus X belongs to M .

- (iv) We show that X belongs to M in general. Let $\ell_i^+ = \max\{\ell_i, 0\}$ and $\ell_i^- = \min\{\ell_i, 0\}$, then $\ell_i = \ell_i^+ + \ell_i^-$. Further, using the relation $x_i^{p_i} = -x_1^{p_1} - \lambda_i x_2^{p_2}$,

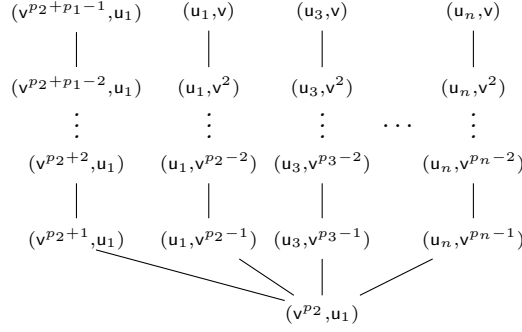
$$X = x_1^{a_1}x_2^{a_2} \prod_{i \geq 3} x_i^{N+\ell_i p_i} = -x_1^{a_1}x_2^{a_2} \prod_{i \geq 3} x_i^{N+\ell_i^- p_i} (x_1^{p_1} + \lambda_i x_2^{p_2})^{\ell_i^+}.$$

This is a linear combination of monomials satisfying (iii), so X belongs to M . \square

Using 5.2 we now translate the modules in 5.4 into ideals.

Proposition 5.5. *With notation in 5.3, up to degree shift, the indecomposable objects in $\text{SCM}^{\mathbb{Z}} S^{\vec{s}}$ are precisely the following ideals of $S^{\vec{s}}$, and furthermore they correspond to the*

dual graph of the minimal resolution of $\text{Spec } S^{\bar{s}}$ (1.F) in the following way:



Proof. We first claim that $S(\vec{x}_1)^{\bar{s}} \cong (v^{p_1+p_2-1}, u_1)$. Indeed, since S is a graded domain, multiplication by any homogeneous element $S \rightarrow S$ is injective. Thus, multiplying by $x_2 \dots x_n$, we see that $S(\vec{x}_1)^{\bar{s}}$ is isomorphic to the $S^{\bar{s}}$ -submodule of S generated by $x_2^{p_1+p_2} x_3^{p_1} \dots x_n^{p_1}$ and $x_1 \dots x_n$, that is generated by u_2 and v . But then

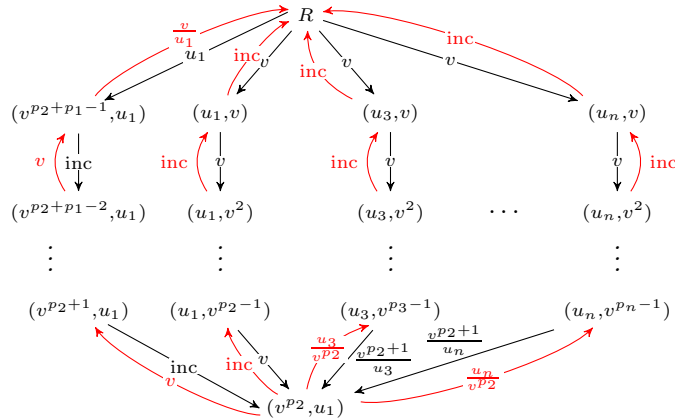
$$u_1(u_2, v) = (u_1 u_2, u_1 v) \stackrel{5.3}{\cong} (v^{p_1+p_2}, u_1 v) = (v^{p_1+p_2-1}, u_1) v,$$

which shows that $S(\vec{x}_1)^{\bar{s}} \cong (v^{p_1+p_2-1}, u_1)$. The other cases are similar. Next,

$$\text{End}_S^{\mathbb{L}}\left(\bigoplus_{\vec{y} \in [0, \vec{c}]} S(\vec{y})\right) \cong \begin{array}{ccccccc} & & S & & & & \\ & \swarrow x_1 & \downarrow x_2 & \downarrow x_3 & \downarrow x_n & \searrow & \\ S(\vec{x}_1) & S(\vec{x}_2) & S(\vec{x}_3) & \dots & S(\vec{x}_n) & & \\ \downarrow x_1 & \downarrow x_2 & \downarrow x_3 & & \downarrow x_n & & \\ S(2\vec{x}_1) & S(2\vec{x}_2) & S(2\vec{x}_3) & & S(2\vec{x}_n) & & \\ \vdots & \vdots & \vdots & & \vdots & & \\ S((p_1-2)\vec{x}_1) & S((p_2-2)\vec{x}_2) & S((p_3-2)\vec{x}_3) & & S((p_n-2)\vec{x}_n) & & \\ \downarrow x_1 & \downarrow x_2 & \downarrow x_3 & & \downarrow x_n & & \\ S((p_1-1)\vec{x}_1) & S((p_2-1)\vec{x}_2) & S((p_3-1)\vec{x}_3) & & S((p_n-1)\vec{x}_n) & & \\ & \swarrow x_1 & \swarrow x_2 & \downarrow x_3 & \swarrow x_n & & \\ & & & S(\vec{c}) & & & \end{array} \quad (5.B)$$

by [GL1], which, after passing through the categorical equivalence $\text{CM}^{\mathbb{L}} S \simeq \text{CM}^{\mathbb{Z}} R$ of 4.8 gives the degree zero part of $\Gamma_{\vec{s}}$. After passing to the completion, as in 4.22 these arrows remain in the quiver of the reconstruction algebra, which by 2.9 forces the positions. \square

Proposition 5.6. *The reconstruction algebra $\Gamma_{\vec{s}}$ is given by the following quiver, where the arrows correspond to the following morphisms*



Proof. Under the isomorphisms in 5.3 and 5.5, the morphisms in (5.B) become

$$\begin{array}{ccccccc}
 & & R & & & & \\
 & \swarrow & & \searrow & & \swarrow & \searrow \\
 & u_1 & & & & & \\
 \swarrow & & \swarrow & \searrow & \swarrow & \searrow & \\
 (v^{p_2+p_1-1}, u_1) & (u_1, v) & (u_3, v) & & (u_n, v) & & \\
 \downarrow \text{inc} & \downarrow v & \downarrow v & & \downarrow v & & \\
 (v^{p_2+p_1-2}, u_1) & (u_1, v^2) & (u_3, v^2) & \cdots & (u_n, v^2) & & \\
 \vdots & \vdots & \vdots & & \vdots & & \\
 (v^{p_2+2}, u_1) & (u_1, v^{p_2-2}) & (u_3, v^{p_3-2}) & & (u_n, v^{p_n-2}) & & \\
 \downarrow \text{inc} & \downarrow v & \downarrow v & & \downarrow v & & \\
 (v^{p_2+1}, u_1) & (u_1, v^{p_2-1}) & (u_3, v^{p_3-1}) & & (u_n, v^{p_n-1}) & & \\
 & \searrow \text{inc} & \searrow v & \swarrow \frac{v^{p_2+1}}{u_3} & \swarrow \frac{v^{p_2+1}}{u_n} & & \\
 & & (v^{p_2}, u_1) & & & &
 \end{array} \tag{5.C}$$

From here, we first work on the completion, which is naturally filtered, then goes back to the graded setting by taking the associated graded ring. We know that the quiver of the reconstruction algebra from (4.T), and we know that for every special CM module X , we must be able to hit the generators of X by composing arrows starting at the vertex R and ending at the vertex corresponding to X , without producing any cycles. Since the arrows in (5.C) are already forced to be arrows in the reconstruction algebra, it remains to choose a basis for the remaining red arrows. For example, the generator v^{p_2+1} in (v^{p_2+1}, u_1) must come from a composition of arrows R to (v^{p_2}, u_1) , followed by the bottom left arrow. Since we can see v^{p_2} as a composition of maps from R to (v^{p_2}, u_1) , this forces the bottom left red arrow to be v . The remaining arrows are similar. \square

Theorem 5.7. *The reconstruction algebra $\Gamma_{\bar{s}}$ is isomorphic to the path algebra of the quiver $\overline{Q}_{\mathbf{p}}$ subject to relations given by*

- (1) *The canonical algebra relations on the black arrows*
- (2) *At every vertex, all 2-cycles that exist at that vertex are equal.*

Proof. This is very similar to [W4, 4.11]. Denote the set of relations in the statement by \mathcal{S}' . Since everything is graded, we first work in the completed case (so we can use [BIRS, 3.4]) and we prove that the completion of reconstruction algebra is given as the completion of kQ (denoted $k\hat{Q}$) modulo the closure of the ideal $\langle \mathcal{S}' \rangle$ (denoted $\overline{\langle \mathcal{S}' \rangle}$). The non-completed version of the theorem then follows by simply taking the associated graded ring of both sides of the isomorphism.

Set $Q := \overline{Q}_{\mathbf{p}, \lambda}$ (as in (4.T)), then by 5.6 there is a natural surjection $\psi: k\hat{Q} \rightarrow \hat{\Gamma}$ with $\mathcal{S}' \subseteq I := \text{Ker } \psi$. Denote the radical of $k\hat{Q}$ by J and further let V denote the set of vertices of Q . Below we show that the elements of \mathcal{S}' are linearly independent in $I/(IJ + JI)$, hence we may extend \mathcal{S}' to a basis \mathcal{S} of $I/(IJ + JI)$. Since \mathcal{S} is a basis, by [BIRS, 3.4(a)] $I = \overline{\langle \mathcal{S} \rangle}$, so it remains to show that $\mathcal{S} = \mathcal{S}'$. But by [BIRS, 3.4(b)]

$$\#(e_a k\hat{Q} e_b) \cap \mathcal{S} = \dim \text{Ext}_{\hat{\Gamma}}^2(S_a, S_b)$$

for all $a, b \in V$, where S_a is the simple module corresponding to vertex a . From 4.22 (i.e. [W2]), this is equal to some number given by intersection theory. Simply inspecting our set \mathcal{S}' and comparing to the numbers in 4.22, we see that

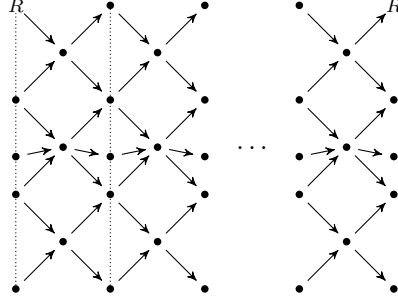
$$\#(e_a k\hat{Q} e_b) \cap \mathcal{S} = \#(e_a k\hat{Q} e_b) \cap \mathcal{S}'$$

for all $a, b \in V$, proving that the number of elements in \mathcal{S} and \mathcal{S}' are the same. Hence $\mathcal{S}' = \mathcal{S}$ and so $I = \overline{\langle \mathcal{S}' \rangle}$, as required.

Thus it suffices to show that the elements of \mathcal{S}' are linearly independent in $I/(IJ + JI)$. This is identical to the proof of [W4, 4.12], so we omit the details. \square

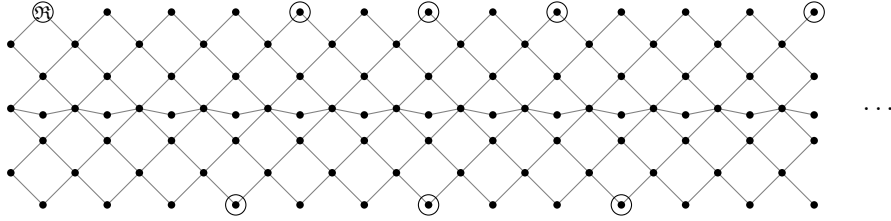
Whilst thinking of the special CM modules as ideals makes everything much more explicit, doing this forgets the grading. Indeed, the reconstruction algebra $\Gamma_{\bar{s}}$ has a natural grading induced from the Veronese construction.

with $m \geq 3$ in 6.1, by [AR] the AR quiver of $\mathfrak{R} \cong k[[x, y]]^{\mathbb{O}_m}$ is



where there are precisely m repetitions of the original \tilde{E}_7 shown in dotted lines. The left and right hand sides of the picture are identified, and there is no twist in this AR quiver. Thus as m increases (and the group $\mathbb{O}_{12(m-2)+1}$ changes), the AR quiver becomes longer.

Regardless of the length m , the special CM \mathfrak{R} -modules always have the following position in the AR quiver



In particular, comparing this to (1.A), we observe the following coincidences.

- (1) The AR quiver of $\mathbf{CM} \mathfrak{R}$ is the quotient of the AR quiver of $\mathbf{vect} \mathbb{X}$ modulo $\tau^{12(m-2)+1} = ((12(m-2) + 1)\vec{\omega})$.
- (2) The canonical tilting bundle \mathcal{E} on \mathbb{X} is given by the circled vertices in (1.A), and so under the identification in (1), this gives the additive generator of $\mathbf{SCM} \mathfrak{R}$.

The same coincidence can also be observed for type \mathbb{T} and \mathbb{I} by replacing 12 by 6 and 30 respectively. To give a theoretical explanation to these observations, we need the following preparation, where recall that $\vec{s} = \sum_{i=1}^3 \vec{x}_i$ and $\vec{\omega} = \vec{c} - \vec{s}$ since $n = 3$.

Lemma 6.2. *Define h as follows*

Type	h
\mathbb{T}	6
\mathbb{O}	12
\mathbb{I}	30

Then $(h+1)\vec{\omega} = -\vec{s}$ and $(h(m-2)+1)\vec{\omega} = -\vec{s}_{m-3}$.

Proof. If $(p_1, p_2, p_3) = (2, 3, 3)$, then $6\vec{\omega} = (6-3-2-2)\vec{c} = -\vec{c}$ and so $7\vec{\omega} = -\vec{s}$. Similarly, if $(p_1, p_2, p_3) = (2, 3, 4)$ then $12\vec{\omega} = (12-6-4-3)\vec{c} = -\vec{c}$, thus $13\vec{\omega} = -\vec{s}$. Lastly, if $(p_1, p_2, p_3) = (2, 3, 5)$ then $30\vec{\omega} = (30-15-10-6)\vec{c} = -\vec{c}$, hence $31\vec{\omega} = -\vec{s}$.

Therefore $(h(m-2)+1)\vec{\omega} = -(m-2)\vec{s} - (m-3)\vec{\omega} = -\vec{s} - (m-3)\vec{c} = -\vec{s}_{m-3}$. \square

Let \mathcal{C} be an additive category with an action by a cyclic group $G = \langle g \rangle \cong \mathbb{Z}$. Assume that, for any $X, Y \in \mathcal{C}$, $\mathrm{Hom}_{\mathcal{C}}(X, g^i Y) = 0$ holds for $i \gg 0$. The *complete orbit category* \mathcal{C}/G has the same object as \mathcal{C} and the morphism sets are given by

$$\mathrm{Hom}_{\mathcal{C}/G}(X, Y) := \prod_{i \in \mathbb{Z}} \mathrm{Hom}_{\mathcal{C}}(X, g^i Y)$$

for $X, Y \in \mathcal{C}$, where the composition is defined in the obvious way.

Theorem 6.3. *Let R be the $(m-3)$ -Wahl Veronese subring associated with $(p_1, p_2, p_3) = (2, 3, 3)$, $(2, 3, 4)$ or $(2, 3, 5)$ and $m \geq 3$, and \mathfrak{R} its completion. Let $G \leq \mathbb{L}$ be the infinite cyclic group generated by $(h(m-2)+1)\vec{\omega} = -\vec{s}_{m-3}$. Then*

(1) *There are equivalences $\mathbf{vect} \mathbb{X} \simeq \mathbf{CM}^{\mathbb{Z}} R$ and*

$$F: (\mathbf{vect} \mathbb{X})/G \xrightarrow{\sim} \mathbf{CM} \mathfrak{R}.$$

(2) *For the canonical tilting bundle \mathcal{E} on \mathbb{X} , we have $\mathbf{SCM} \mathfrak{R} = \text{add } F\mathcal{E}$.*

Proof. Since $(h(m-2)+1)\vec{\omega} = -\vec{s}_{m-3}$ is a non-zero element in $-\mathbb{L}_+$, for any $X, Y \in \mathbf{vect} \mathbb{X}$, necessarily $\text{Hom}_{\mathbb{X}}(X, Y(i(h(m-2)+1)\vec{\omega})) = 0$ holds for $i \gg 0$. Therefore the complete orbit category $(\mathbf{vect} \mathbb{X})/G$ is well-defined.

(1) There are equivalences $\mathbf{vect} \mathbb{X} \simeq \mathbf{CM}^{\mathbb{L}} S \simeq \mathbf{CM}^{\mathbb{Z}} R$, where the first equivalence is standard [GL1], and the second is 4.8. Furthermore, the following diagram commutes.

$$\begin{array}{ccc} \mathbf{vect} \mathbb{X} & \longrightarrow & \mathbf{CM}^{\mathbb{Z}} R \\ (\vec{s}_{m-3}) \downarrow & & \downarrow (1) \\ \mathbf{vect} \mathbb{X} & \longrightarrow & \mathbf{CM}^{\mathbb{Z}} R \end{array}$$

Since \mathfrak{R} has only finitely many indecomposable CM modules (see e.g. [Y, 15.14]), there is an equivalence $(\mathbf{CM}^{\mathbb{Z}} R)/\mathbb{Z} \simeq \mathbf{CM} \mathfrak{R}$. Therefore $(\mathbf{vect} \mathbb{X})/G \simeq (\mathbf{CM}^{\mathbb{Z}} R)/\mathbb{Z} \simeq \mathbf{CM} \mathfrak{R}$.

(2) This follows by the equivalences in (1), the definition of \mathcal{E} , and 3.10. \square

As one final observation, recall that for a canonical algebra Λ , the *preprojective algebra* of Λ is defined by

$$\Pi := \bigoplus_{i \geq 0} \Pi_i, \quad \Pi_i := \text{Hom}_{\mathbf{D}^b(\text{mod } \Lambda)}(\Lambda, \tau^{-i} \Lambda),$$

where τ is the Auslander-Reiten translation in the derived category $\mathbf{D}^b(\text{mod } \Lambda)$. Moreover, for a positive integer t , we denote the t -th Veronese subring of Π by

$$\Pi^{(t)} := \bigoplus_{i \geq 0} \Pi_{ti}.$$

As notation we write Γ_m for the reconstruction algebra of R , where R is the $(m-3)$ -Wahl Veronese subring associated with $(p_1, p_2, p_3) = (2, 3, 3)$, $(2, 3, 4)$ or $(2, 3, 5)$ and $m \geq 3$, which correspond to one of the types \mathbb{T} , \mathbb{O} or \mathbb{I} in 6.1.

The following is an analogue of 4.23, but also describes the other graded pieces.

Proposition 6.4. *There is an isomorphism of \mathbb{Z} -graded algebras*

$$\Pi^{(h(m-2)+1)} \cong \Gamma_m.$$

Proof. By 6.2 we know that $(h(m-2)+1)\vec{\omega} = -\vec{s}_{m-3}$. Setting $M = \bigoplus_{\vec{y} \in [0, \vec{c}]} S(\vec{y})$, then $\Pi_i^{(h(m-2)+1)}$ for $i \geq 0$ is given by

$$\begin{aligned} \text{Hom}_{\mathbf{D}^b(\text{mod } \Lambda)}(\Lambda, \tau^{-(h(m-2)+1)i} \Lambda) &\cong \text{Hom}_{\mathbb{L}_S}(M, M(-i(h(m-2)+1)\vec{\omega})) \\ &\cong \text{Hom}_{\mathbb{L}_S}(M, M(i\vec{s}_{m-3})) \\ &\cong (\Gamma_m)_i. \end{aligned}$$

Thus all the graded pieces match. It is easy to see that the isomorphisms are natural, and so give an isomorphism of graded rings. \square

Remark 6.5. By 6.4, it follows that in fact on the abelian level

$$\mathbf{qgr}^{\mathbb{Z}} \Gamma_m \simeq \mathbf{qgr}^{\mathbb{Z}} \Pi^{(h(m-2)+1)}$$

and so, combining 4.8 and 4.25,

$$\mathbf{coh} \mathbb{X} \simeq \mathbf{qgr}^{\mathbb{Z}} \Pi^{(h(m-2)+1)}$$

for any $m \geq 3$. This is a stronger version of results of [GL1] and Minamoto [M], which combine to say that for the weighted projective lines of non-tubular type there are derived equivalences

$$\mathbf{D}^b(\mathbf{coh} \mathbb{X}) \simeq \mathbf{D}^b(\text{mod } \Lambda) \simeq \mathbf{D}^b(\mathbf{qgr}^{\mathbb{Z}} \Pi).$$

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